EIS property for dependence spaces

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Introduction

Dependent and independent sets

General properties

Steinitz’ exchange theorem

EIS property

Bibliography
According to F. Gécseg, H. Jürgensen [3] the result which is usually referred to as the ”Exchange Lemma” states that for transitive dependence, every independent set can be extended to form a basis. In [5] we discussed some interplay between notions discussed in [2], [3] and [6], [7]. Another proof was presented there, of the result of N.J.S. Hughes [6] on Steinitz’ exchange theorem for infinite bases in connection with the notions of transitive dependence, independence and dimension as introduced in [6], [7]. In that proof we assumed Kuratowski-Zorn’s Lemma of [11], [12] as a requirement pointed in [6]. Later, in dependence spaces we extended the results to EIS property known in general algebra as Exchange of Independent Sets Property.
We use a modification of the the notation of [6]–[7]:
\( a, b, c, \ldots, x, y, z, \ldots \) (with or without suffices) to denote the
elements of a space \( S \) and \( A, B, C, \ldots, X, Y, Z, \ldots, \) for subsets of \( S \).
\( \Delta, S \) denote a family of subsets of \( S \), \( n \) is always a positive integer.
\( A \cup B \) denotes the union of sets \( A \) and \( B \), \( A + B \) denotes the
disjoint union of \( A \) and \( B \), \( A - B \) denotes the difference of \( A \) and \( B \), i.e. is the set of those elements of \( A \) which are not in \( B \).
The following definitions are due to N.J.S. Hughes, invented in 1962 in [6], if there is defined a set $\Delta$, whose members are finite subsets of $S$, each containing at least 2 elements:

**Definition**

A set $A$ is called *directly dependent* if $A \in \Delta$. 
Definition
An element $x$ is called \textit{dependent on} $A$ and is denoted by $x \sim \Sigma A$ if either $x \in A$ or if there exist distinct elements $x_0, x_1, \ldots, x_n$ such that

\begin{equation}
\{x_0, x_1, \ldots, x_n\} \in \Delta
\end{equation}

where $x_0 = x$ and $x_1, \ldots, x_n \in A$

and \textit{directly dependent} on $\{x\}$ or $\{x_1, \ldots, x_n\}$, respectively.
Definition
A set $A$ is called *dependent* (with respect to $\Delta$) if (1) is satisfied for some distinct elements $x_0, x_1, ..., x_n \in A$. Otherwise $A$ is *independent*.

Definition
$A$ is called a *basis of* $S$, if a set $A$ is a subset of $S$ which is *independent* and for any $x \in S$, $x \sim \Sigma A$, i.e. every element $x$ of $S$ is dependent on $A$. 
A similar definition of a *dependence D* was introduced in [3]. In the paper authors based on the theory of dependence in universal algebras as outlined in [2]. We accept the well known:

**Definition**
The *span* $\langle X \rangle$ of a subset $X$ of $S$ is the set of all elements of $S$ which depends on $X$, i.e. $x \in \langle X \rangle$ iff $x \sim \Sigma X$.

**Definition**
**TRANSITIVITY AXIOM:**

If $x \sim \Sigma A$ and for all $a \in A$, $a \sim \Sigma B$, then $x \sim \Sigma B$. 
Definition
A set $S$ is called a *dependence space* if there is defined a set $\Delta$, whose members are finite subsets of $S$, each containing at least 2 elements, and if the Transitivity Axiom is satisfied.
Since then $S$ will always be a dependence space satisfying the transitivity axiom.
Remark
Equivalently, transitivity was defined in [3] in the following way: the dependence is said to be transitive if $<X > = << X >>$ for every subset $X$ of $S$.

Transitive dependence systems have been studied under several different names (cf. [13], p. 7). The collection of independent sets of a dependent space on a finite set is known as a matroid [13]. The basic results on the interplay between algebraic closure operators with exchange property of [2] and (transitive) dependence spaces were formulated as conditions (1) and (2) in Theorem 3.8 and conditions (1) – (3) of Lemma 3.9 of [3].
Given a transitive dependence space $S$. One may consider the operator $<>$ on subsets $X$ of $S$ as a generalized closure operator, i.e. extensive, monotone and idempotent mapping (cf. e.g. Birkhoff [1]). Obviously, by the definition, the closure operator $<>$ in $S$ has finite character (see [10], p. 647), i.e. for every subset $X$ of $S$ it satisfies the property:

(F) $<X> = \bigcup <F>$, where $F$ runs over the family of all finite subsets of $X$. 

In Linear Algebra, Steinitz’ exchange Lemma states that:
if \( a \in < A \cup \{b\} > \) and \( a \notin < A > \), then \( b \in < A \cup \{a\} > \).
In particular, if \( A \) is independent and \( a \notin < A > \), then:
\( A \cup \{a\} \) is independent.
The following lemma is a generalization of the result of P.M. Cohn [2] (cf. the property (E) of [10], p. 206, called there an exchange of an independent sets or Theorem 3.8 of [3], p. 426):

**Lemma**

*In a dependence space $S$, assume that:*

$a \in < A \cup \{b\}>$ and $a \notin < A >$.

*Then* $b \in < A \cup \{a\}>$. 

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EIS property for dependence spaces
Proof
If $a \in <A \cup \{b\}> - <A>$, then there exists $a_1, ..., a_n \in A$, such that $a \sim \Sigma\{b, a_1, ..., a_n\}$, i.e. $\{a, a_1, ...a_n, b\} \in \Delta$.
Therefore $b \in <\{a\} \cup A>$. □

It is clear, that for an independent set $A$, one gets for each $a \in A$, that $a \not\in <A - \{a\}>$. 
The EIS (exchange of independent sets) property was introduced by A. Hulanicki, E. Marczewski, E. Mycielski in [8]. First we recall their original definition of EIS property (see [8], [10] p. 647–659). In their paper they use the terminology and notation of [9] (with slight modifications). An abstract algebra is a (nonempty) set with a family of fundamental finitary operations. For any nonempty set \( E \subset A \), \( C(E) \) denotes the subalgebra generated by \( E \), \( C(\emptyset) \) is denoting the set of algebraic constants (i.e. the values of the constant algebraic operations).
The operation $C$ has finite character, i.e. for every $E \subset A$, the following holds:

$$ (2) \quad C(E) = \bigcup C(F), $$

where $F$ runs over the family of all finite subsets of $F$ of $E$.

The following theorem about exchange of independent sets is true for all algebras (see [9], p. 58, theorem 2.4 (ii)): 
Theorem (E. Marczewski)

Let $P, Q$ and $R$ be subsets of an algebra. If

(3) $P \cup Q$ is independent,
(4) $P \cap Q = \emptyset$,
(5) $R$ is independent,
(6) $C(R) = C(Q),$

then $P \cup R$ is independent.
As the authors of [8] noticed, it might seem at first glance that the relation $C(R) = C(Q)$ could be replaced by a weaker one: $R \subset C(Q)$. Since, as it can be seen from the results of [8] p. 204, this is not generally true, the authors say that an algebra satisfies the condition of exchange of independent sets (EIS) whenever for any subsets $P$, $Q$ and $R$ of it, the relations: $P \cup Q$ is independent, $P \cap Q = \emptyset$, $R$ is independent and $R \subset C(Q)$ imply that $P \cup R$ is independent.
It seems to be worth of mentioning, that the results of the paper [8] have been presented without proofs by E. Marczewski in his lecture *Independence in abstract algebras. Result and problems* to the Conference on General Algebra, held in Warsaw, September 11-17, 1964.

In the 70ties several algebraists were dealing with problems of the satisfaction of EIS-property in several algebras. Among them, one of the first were: A. Hulnicki, W. Narkiewicz, Jerzy Płonka, J. Schmidt, S. Świerczkowski, Tadeusz Traczyk and many others. Some problems were announced in the New Scottish Book at Wrocław University.
We transform the original definition of EIS property from *algebras* to *dependence spaces* in the natural way:

**Definition**

A dependence space $S$ satisfies the EIS property, if for arbitrary subsets $P$, $Q$ and $R$ of $S$ the conditions:

(7) $P \cap Q = \emptyset$;
(8) $P \cup Q$ is an independent set in $S$;
(9) $R$ is an independent set in $S$, $R \subseteq \prec Q \succ$;
altogether imply that:
(10) $P \cup R$ is an independent set.
Theorem

*In a dependence space $S$, the EIS property holds.*
Proof
Assume (7) – (9).
To show (10) assume a contrario that \( P \cup R \) is a dependent set. Therefore there exist (all different) elements \( a_1, \ldots, a_n, b_1, \ldots, b_m \in P \cup R \) with \( a_1, \ldots, a_n \in P \) and \( b_1, \ldots, b_m \in R \) and such that \( \{a_1, \ldots, a_n, b_1, \ldots, b_m\} \in \Delta \). From (7) and (9) it follows that there exists an element \( a_1 \in P \) such that \( a_1 \sim \Sigma \{a_2, \ldots, a_n, b_1, \ldots, b_m\} \), i.e. \( a_1 \sim \Sigma ((P \setminus \{a_1\}) \cup R) \). But for every element \( b \in R \), \( b \sim \Sigma Q \), therefore \( b \sim \Sigma ((P \setminus \{a_1\}) \cup Q) \).
Moreover, \( c \sim \Sigma ((P \cup Q) \setminus \{a_1\}) \), for every \( c \in ((P \setminus \{a_1\}) \cup R) \). Thus, by the transitivity axiom \( a_1 \sim \Sigma ((P \cup Q) \setminus \{a_1\})) \). That contradicts (8), as \( a_1 \in P \cup Q \) and it is clear, that for an independent set \( A \), one gets for each \( a \in A \), that \( a \notin A \setminus \{a\} \).
\( \Box \)
Remark

The theorem above may be easily proved via THEOREM of [8] (see [8], p. 207 or [10], p. 651), which states, that if a generalized closure operator (here $\langle\rangle$) has finite character and Steinitz exchange property, then it satisfies the condition of exchange of independent sets.


