Finite groups with some $CEP$-subgroups

Izabela Agata Malinowska

Institute of Mathematics
University of Białystok, Poland

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A subgroup $H$ of a group $G$ satisfies the *Congruence Extension Property* in $G$ (or $H$ is a *CEP-subgroup* of $G$) if whenever $N$ is a normal subgroup of $H$, there is a normal subgroup $L$ of $G$ such that $N = H \cap L$. 

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A subgroup $H$ of a group $G$ is an *NR-subgroup* of $G$ (*Normal Restriction*) if, whenever $N \trianglelefteq H$, $N^G \cap H = N$, where $N^G$ is the *normal closure* of $N$ in $G$ (the smallest normal subgroup of $G$ containing $N$).
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A subgroup $H$ of a group $G$ is *normal sensitive* in $G$ if the following holds:

$$\{ N \mid N \text{ is normal in } H \} = \{ H \cap L \mid L \text{ is normal in } G \}.$$
A group $G$ is **nilpotent** if it has a **central series**, that is, a normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that $G_{i+1}/G_i$ is contained in the centre of $G/G_i$ for all $i$. 

**Example**

$S_3$ is a supersoluble group that is not nilpotent.

$A_4$ is a soluble group that is not supersoluble.
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A subgroup $H$ of a group $G$ is a *Hall subgroup* of $G$ if

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Let $p$ be a prime. A group $G$ is *$p$-nilpotent* if it has a normal Hall $p'$-subgroup.
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Every nilpotent group is $p$-nilpotent; conversely a group which is $p$-nilpotent for all $p$ is nilpotent.
Basic concepts

Example

\[ H = \langle (12)(34) \rangle \triangleleft V_4 = \langle (12)(34), (13)(24) \rangle \triangleleft A_4 \]
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Let \( G \) be a group. A subgroup \( K \) of \( G \) is **subnormal** in \( G \) if there are a non-negative integer \( r \) and a series

\[ K = K_0 \triangleleft K_1 \triangleleft K_2 \triangleleft \cdots \triangleleft K_r = G \]

of subgroups of \( G \).
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Theorem

Let \( G \) be a group. Then the following properties are equivalent:

1. \( G \) is nilpotent;
2. every subgroup of \( G \) is subnormal;
3. \( G \) is the direct product of its Sylow subgroups.
A group $G$ is *Dedekind* if every subgroup of $G$ is normal in $G$. 

**Theorem (R. Dedekind, 1896)**

A group $G$ is Dedekind if and only if $G$ is abelian or $G$ is a direct product of the quaternion group $Q_8$ of order 8, an elementary abelian $2$-group and an abelian group of odd order.

A subgroup $H$ of a group $G$ is permutable in a group $G$ if $HK = KH$ whenever $K \leq G$.

Let $G$ be a group. If $N \triangleleft G$, then $N$ is permutable in $G$.

**Example**

Let $p$ be an odd prime and let $G$ be an extraspecial group of order $p^3$ and exponent $p^2$. $G$ has all subgroups permutable, but $G$ has non-normal subgroups.
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Theorem (O. Ore, 1939)

If $H$ is a permutable subgroup of a group $G$, then $H$ is subnormal in $G$. 

A group $G$ is an Iwasawa group if every subgroup of $G$ is permutable in $G$.

Theorem (K. Iwasawa, 1941)

Let $p$ be a prime. A $p$-group $G$ is an Iwasawa group if and only if $G$ is a Dedekind group, or $G$ contains an abelian normal subgroup $N$ such that $G/N$ is cyclic and so $G = \langle x \rangle N$ for an element $x$ of $G$ and $x = a + p^s$ for all $a \in N$, where $s \geq 1$ and $s \geq 2$ if $p = 2$. 

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A subgroup of a group $G$ is \textit{s-permutable} in $G$ if it permutes with all Sylow subgroups of $G$. 

Theorem (O.H. Kegel, 1962) 
If $H$ is an $s$-permutable subgroup of $G$, then $H$ is subnormal in $G$. 

Example 
The dihedral group $D_8$ of order 8 has subgroups which are not permutable but all its subgroups are obviously $s$-permutable.
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Let $G$ be a group and let $\alpha$ be an automorphism of $G$. We say that $\alpha$ is a *power automorphism* of $G$ if for every $g \in G$ there exists an integer $n(g)$ such that $n^\alpha = g^{n(g)}$. In other words, $\alpha$ is a power automorphism of $G$ if $\alpha$ fixes all the subgroups of $G$. 

Definition

A group $G$ is a *T-group* if every subnormal subgroup of $G$ is normal in $G$. 

Examples of T-groups: Dedekind groups = nilpotent T-groups; simple groups.
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A group $G$ is a *$T$-group* if every subnormal subgroup of $G$ is normal in $G$. 
Characterizations based on the normal structure

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**Examples of $T$-groups:**

- Dedekind groups = nilpotent $T$-groups;
- simple groups.
Theorem (W. Gaschütz, 1957)

A group $G$ is a soluble $T$-group if and only if the following conditions are satisfied:

1. the nilpotent residual $L$ of $G$ is an abelian Hall subgroup of odd order;
2. $G$ acts by conjugation on $L$ as a group of power automorphisms, and
3. $G/L$ is a Dedekind group.
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Definition

A group $G$ is said to be a $PT$-group when if $H$ is a permutable subgroup of $K$ and $K$ is a permutable subgroup of $G$, then $H$ is a permutable subgroup of $G$. 

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- $T$-groups;
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A group $G$ is a **PST-group** when if $H$ is an $s$-permutable subgroup of $K$ and $K$ is an $s$-permutable subgroup of $G$, then $H$ is an $s$-permutable subgroup of $G$. 

**Examples of PST-groups:** nilpotent groups; PT-groups.

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A group $G$ is a $PST$-group when if $H$ is an $s$-permutable subgroup of $K$ and $K$ is an $s$-permutable subgroup of $G$, then $H$ is an $s$-permutable subgroup of $G$.

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Examples of \textit{PST}-groups:

- nilpotent groups;
- $PT$-groups.

The \textit{PST}-groups are exactly the groups in which every subnormal subgroup is $s$-permutable.
Theorem (R.K. Agrawal, 1975)

Let $G$ be a group with nilpotent residual $L$. The following statements are equivalent:

1. $L$ is an abelian Hall subgroup of odd order in which $G$ acts by conjugation as a group of power automorphisms;
2. $G$ is a soluble PST-group.

Corollary

Let $G$ be a group.

1. $G$ is a soluble PT-group if and only if $G$ is a soluble PST-group whose Sylow subgroups are Iwasawa groups;
2. $G$ is a soluble T-group if and only if $G$ is a soluble PST-group whose Sylow subgroups are Dedekind groups.
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Theorem (R.K. Agrawal, 1975)

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Every soluble PST-group is supersoluble.
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Example

\( S_3 \times S_3 \) is a supersoluble group which is not a PST-group.
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Example

$S_3 \times S_3$ is a supersoluble group which is not a PST-group.

The classes of all soluble $T$, $PT$- and $PST$-groups are closed under taking subgroups.
In the soluble universe:

<table>
<thead>
<tr>
<th>$T$</th>
<th>$PT$</th>
<th>$PST$</th>
<th>supersoluble</th>
</tr>
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<tbody>
<tr>
<td>$\cup_{\mathfrak{ln}}$</td>
<td>$\cup_{\mathfrak{ln}}$</td>
<td>$\cup_{\mathfrak{ln}}$</td>
<td></td>
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<tr>
<td>Dedekind</td>
<td>Iwasawa</td>
<td>nilpotent</td>
<td></td>
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</tbody>
</table>

$$T \subset PT \subset PST \subset \text{supersoluble}$$
A group $H$ of a group $G$ is a **CEP-subgroup** of $G$ if whenever $N$ is a normal subgroup of $H$, there is a normal subgroup $L$ of $G$ such that $N = H \cap L$.

**Theorem (S. Bauman, 1974)**

*Every subgroup of a group $G$ is a CEP-subgroup of $G$ if and only if $G$ is a soluble $T$-group.*
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**Theorem (S. Bauman, 1974)**

Every subgroup of a group $G$ is a CEP-subgroup of $G$ if and only if $G$ is a soluble $T$-group.

**Theorem (I.A.M., 2012)**

A group $G$ is a soluble $T$-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is a CEP-subgroup of $G$. 
Let $p$ be a prime. A group $G$ satisfies the property $\text{CEP}_p$ if a Sylow $p$-subgroup of $G$ is a $\text{CEP}$-subgroup of $G$. 
Local characterizations

Let \( p \) be a prime. A group \( G \) satisfies \textit{the property CEP}_p if a Sylow \( p \)-subgroup of \( G \) is a CEP-subgroup of \( G \).

\begin{center}
\textbf{Theorem (I.A.M. 2013)}
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A group \( G \) is a soluble PST-group if and only if every subgroup of \( G \) satisfies CEP\(_p\) for all \( p \in \pi(G) \).

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**Theorem (I.A.M. 2014)**

Let $G$ be a group. The following conditions are equivalent:

1. $G$ is a soluble PT-group;
2. $G$ satisfies $\text{CEP}_p$ and $G$ has Iwasawa Sylow $p$-subgroups for every $p \in \pi(G)$. 
Theorem (I.A.M., 2013)

If all proper subgroups of even order of a group $G$ satisfy $CEP_p$ for every $p$, then $G$ is either 2-nilpotent or minimal non-nilpotent. In particular, $G$ is soluble.

Theorem (S. Li, Y. Zhao, 1988)

Let $G$ be a non-soluble group. Assume that soluble subgroups of $G$ are either 2-nilpotent or minimal non-nilpotent. Then $G$ is one of the following groups:

1. $PSL(2, 2^f)$, where $f$ is a positive integer such that $2^f - 1$ is a prime;
2. $PSL(2, q)$, where $q$ is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod{8}$;
3. $SL(2, q)$, where $q$ is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod{8}$.
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Theorem (I.A.M., 2012)

Let $G$ be a group all of whose second maximal subgroups of even order are soluble PST-groups. Then $G$ is either a soluble group or one of the following groups:

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2. $\text{PSL}(2, p)$, where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;
3. $\text{PSL}(2, 3^f)$, where $f$ is an odd prime;
4. $\text{SL}(2, 3^f)$, where $f$ is an odd prime and $(3^f - 1)/2$ is a prime;
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Bibliography:

Thank you