On a general construction of countable universal homogeneous algebraic systems

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Homogeneous structures

\[ \mathcal{A} \]

automorphism

isomorphism
Fraïssé theory

**age**($\mathcal{A}$) — the class of all finitely generated struct’s which embed into $\mathcal{A}$

*amalgamation class* — a class $\mathbf{K}$ of fin. generated struct’s s.t.
- there are only countably many pairwise noniso struct’s in $\mathbf{K}$;
- $\mathbf{K}$ has (HP);
- $\mathbf{K}$ has (JEP); and
- $\mathbf{K}$ has (AP):
  for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}$ and embeddings $f : \mathcal{A} \hookrightarrow \mathcal{B}$ and $g : \mathcal{A} \hookrightarrow \mathcal{C}$, there exist $\mathcal{D} \in \mathbf{K}$ and embeddings $u : \mathcal{B} \hookrightarrow \mathcal{D}$ and $v : \mathcal{C} \hookrightarrow \mathcal{D}$ such that $u \circ f = v \circ g$. 
Fraïssé theory

**Theorem.** [Fraïssé, 1953]

1. If \( \mathcal{A} \) is a countable homogeneous structure, then \( \text{age}(\mathcal{A}) \) is an amalgamation class.

2. If \( \mathbf{K} \) is an amalgamation class, then there is a unique (up to isomorphism) countable homogeneous structure \( \mathcal{A} \) such that \( \text{age}(\mathcal{A}) = \mathbf{K} \).

3. If \( \mathcal{B} \) is a countable structure younger than \( \mathcal{A} \) (that is, \( \text{age}(\mathcal{B}) \subseteq \text{age}(\mathcal{A}) \)), then \( \mathcal{B} \hookrightarrow \mathcal{A} \).

**Definition.** If \( \mathbf{K} \) is an amalgamation class and \( \mathcal{A} \) is the countable homogeneous structure such that \( \text{age}(\mathcal{A}) = \mathbf{K} \), we say that \( \mathcal{A} \) is the *Fraïssé limit* of \( \mathbf{K} \).
Some prominent Fraïssé limits

$(\mathbb{Q}, <)$ — the Fraïssé limit of the class of all linear orders

$\mathcal{U}_\mathbb{Q}$ — Fraïssé limit of the class of finite metric spaces with rational distances (the rational Urysohn space)

$\mathcal{R}$ — Fraïssé limit of the class of all finite graphs (the Rado graph)

$\mathcal{P}$ — Fraïssé limit of the class of all finite posets (the random poset)
The Urysohn space

P. URYSOHN: *Sur un espace métrique universel.*

$\mathcal{U}$ — complete separable metric space which is homogeneous
and embeds all separable metric spaces.

\[
\mathcal{U} = \overline{\mathcal{U}_\mathbb{Q}}
\]
M. Katětov: *On universal metric spaces.*
General topology and its relations to modern analysis and algebra. VI (Prague, 1986),

A Katětov function over a finite rational metric space $X$ is every function $\alpha : X \rightarrow \mathbb{Q}$ such that

$$|\alpha(x) - \alpha(y)| \leq d(x, y) \leq \alpha(x) + \alpha(y)$$

$K(X)$ is all Katětov functions over $X$, which is a rational metric space under sup metric

$$\text{colim}(X \hookrightarrow K(X) \hookrightarrow K^2(X) \hookrightarrow K^3(X) \hookrightarrow \cdots ) = \mathcal{U}_\mathbb{Q}$$
M. Katětov: *On universal metric spaces.*

**Observation 1.** $K(X)$ is the set of all 1-types over $X$ (in an appropriate first-order language).

**Observation 2.** $K$ is functorial.
Katětov functors

\( \mathcal{A} \) — a category of fin generated \( L \)-struct’s with (HP) and (JEP)

\( \mathcal{C} \) — the category of all colimits of \( \omega \)-chains in \( \mathcal{A} \)

**Definition.** A functor \( K : \mathcal{A} \to \mathcal{C} \) is a *Katětov functor* if \( K \) preserves embeddings and there exists a natural transformation \( \eta : \text{ID} \to K \) such that for every embedding \( f : A \hookrightarrow B \) in \( \mathcal{A} \) where \( B \) is a 1-point extension of \( A \) there is an embedding \( g : B \hookrightarrow K(A) \) satisfying

\[
\begin{array}{c}
A \\
\downarrow^{\eta_A}
\end{array} \quad \begin{array}{c}
\to
\end{array} \quad \begin{array}{c}
K(A)
\end{array}
\]
A Katětov functor exists for the following categories $\mathcal{A}$:
- finite linear orders with order-preserving maps,
- finite graphs with graph homomorphisms,
- finite $K_n$-free graphs with embeddings,
- finite digraphs with digraph homomorphisms,
- finite rational metric spaces with nonexpansive maps,
- finite posets with order-preserving maps,
- finite boolean algebras with homomorphisms,
- finite semilattices with embeddings,
- finite lattices with embeddings,
- finite distributive lattices with embeddings.

A Katětov functor does not exist for the category of finite $K_n$-free graphs and graph homomorphisms.
Existence of Katétov functors

\( \mathcal{A} \) — a category of fin generated \( L \)-struct’s with (HP) and (JEP)

\( \mathcal{C} \) — the category of all colimits of \( \omega \)-chains in \( \mathcal{A} \)

**Theorem.** If there exists a Katétov functor \( K : \mathcal{A} \to \mathcal{C} \), then \( \mathcal{A} \) is an amalgamation class, and its Fraïssé limit \( F \) can be obtained by the “Katétov construction” starting from an arbitrary \( A \in \mathcal{A} \):

\[
F = \text{colim}(X \hookrightarrow K(A) \hookrightarrow K^2(A) \hookrightarrow K^3(A) \hookrightarrow \cdots).
\]
Katětov functors for categories of algebras

$L$ — algebraic language

$\mathcal{V}$ — a variety of $L$-algebras understood as a category of $L$-algebras with embeddings

$\mathcal{A}$ — the full subcategory of $\mathcal{V}$ spanned by all finitely generated algebras in $\mathcal{V}$

$\mathcal{C}$ — the full subcategory of $\mathcal{V}$ spanned by all countably generated algebras in $\mathcal{V}$

**Theorem.** Suppose that there are only countably many isomorphism types in $\mathcal{A}$. There exists a Katětov functor $K : \mathcal{A} \rightarrow \mathcal{C}$ if and only if $\mathcal{A}$ is the amalgamation class.
The Importance of Being Earnest Functor

**Theorem.** Let $K : A \to C$ be a Katétov functor and let $F$ be the Fraïssé limit of $A$. Then for every object $C$ in $C$:

- $\text{Aut}(C) \hookrightarrow \text{Aut}(F)$;
- $\text{End}_C(C) \hookrightarrow \text{End}_C(F)$.

**Corollary.** For the following Fraïssé limits $F$ we have that $\text{End}(F)$ embeds all transformation monoids on a countable set:

- $\mathbb{Q}$,
- the random graph [Bonato, Delić, Dolinka 2010],
- the random digraph,
- the rational Urysohn space,
- the random poset [Dolinka 2007],
- the countable atomless boolean algebra.
Corollary. For the following Fraïssé limits $F$ we have that $\text{Aut}(F)$ embeds all permutation groups on a countable set:

- $\mathbb{Q}$ [Truss],
- the random graph [Henson 1971],
- Henson graphs [Henson 1971],
- the random digraph,
- the rational Urysohn space [Uspenskij 1990],
- the random poset,
- the countable atomless boolean algebra,
- the random semilattice,
- the random lattice,
- the random distributive lattice.