The Clone of Compatible Functions of Special Expanded Groups

Nebojša Mudrinski

Institute of Algebra
Johannes Kepler University Linz
Austrian Science Fund FWF P24077
Department of Mathematics and Informatics
University of Novi Sad

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Generation of compatible functions
Expanded groups

An algebra \((A, +, -, 0, F)\) is called an **expanded group** if \((A, +, -, 0)\) is a group and \(F\) is a set of operations on \(A\).

**Examples**

Rings, modules, vector spaces
Expanded Groups

**Expanded groups**

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Definition

An ideal of expanded group $\mathbf{V} = (V, +, -, 0, F)$ is a normal subgroup $I$ of the group $(V, +, -, 0)$ such that $f(a + i) - f(a) \in I$, for all $k \in \mathbb{N}$, all $k$-ary fundamental operations $f \in F$ and all $a \in V^k$, $i \in I^k$.

Proposition

Let $V$ be an expanded group and let $I \in \text{Id} V$. Then

$$\gamma_V(I) := \{(v_1, v_2) \in V^2 | v_1 - v_2 \in I\}$$

is an isomorphism from $(\text{Id} V, \cap, +)$ to $(\text{Con} V, \wedge, \vee)$.
Ideals and Congruences

**Definition**

An **ideal** of expanded group $V = (V, +, −, 0, F)$ is a normal subgroup $I$ of the group $(V, +, −, 0)$ such that $f(a + i) − f(a) ∈ I$, for all $k ∈ \mathbb{N}$, all $k$-ary fundamental operations $f ∈ F$ and all $a ∈ V^k$, $i ∈ I^k$.

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Definition

A function \( f \in V^V \), \( k \in \mathbb{N} \), is compatible on an expanded group \( V \) if for every ideal \( I \in \text{Id} V \) and for every \( a, b \in V^k \) such that \( a \equiv_I b \), we have that \( f(a) \equiv_I f(b) \).

The clone of compatible functions

All compatible functions of an expanded group \( V \) form a clone. We denote it by \( Comp(V) \).

Remark

\( Comp(V) = Polym(Con V) \)
The Clone of Compatible Functions

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**Remark**

\[ \text{Pol}(V) \subseteq \text{Comp}(V). \]

**Theorem (Lausch, Nöbauer, 1976)**

\[ \text{Pol}(\mathbb{Z}_4 \times \mathbb{Z}_2) \neq \text{Comp}(\mathbb{Z}_4 \times \mathbb{Z}_2). \]

**Remark**

If an expanded group \( V \) is of a finite type then \( \text{Pol}(V) \) is finitely generated.

**Question**

For an expanded group \( V \), is \( \text{Comp}(V) \) finitely generated?

**Theorem (E. Aichinger, 2002)**

\( \text{Comp}(\mathbb{Z}_4 \times \mathbb{Z}_2) \) is not finitely generated.
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Compatible Functions and Polynomials

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Characterization for finite generation of the clone of compatible functions of $p$-groups
The lattice of normal subgroups of a $p$-group $G$:

Elements $S_1, \ldots, S_n$ split the lattice of normal subgroups of $G$. 
Definition (Splitting pair)

Let \( V \) be an expanded group. We say that the lattice \( \text{Id} V \) splits if there exists a pair \( (A, B) \in (\text{Id} V)^2 \), such that

\[
A < 1 \land B > 0 \land (\forall X \in \text{Id} V)(X \leq A \lor X \geq B).
\]

Definition (Splitting element)

A splits the lattice \( \text{Id} V \) if \( (\forall X \in \text{Id} V)(X \leq A \lor X \geq A) \).

Remark

If \( \text{Id} V \) has a proper splitting element (distinct from 0 and 1) then it has a splitting pair.
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If $\text{Id } V$ has a proper splitting element (distinct from 0 and 1) then it has a splitting pair.
Theorem (E. Aichinger, M. Lazić, N.M., 2013)

Let \( \{e\} = S_0 < S_1 < \cdots < S_n = G \) be all the elements that split the lattice of normal subgroups of a \( p \)-group \( G \). Then \( \text{Comp}(G) \) is finitely generated iff for every \( i \in \{1, \ldots, n\} \) we have \( S_{i-1} < S_i \) or \( I[S_{i-1}, S_i] \) does not have a splitting pair.
Solution for expanded groups with AP property
Definition (Coalesced ordered sum)

Let $V$ be an expanded group. $\text{Id } V$ is a coalesced ordered sum of intervals $I[0, S_1], \ldots, I[S_n, 1]$ if $\text{Id } V = I[0, S_1] \cup \cdots \cup I[S_n, 1]$, all the intervals are simple and for all $i \in \{1, \ldots, n\}$ $S_i$ splits the lattice $\text{Id } V$. 
AP Property

**Definition**

Let $V$ be an expanded group. $\text{Id} V$ has the **AP property** if each two prime intervals $I[A, B]$ and $I[A, C]$ of $\text{Id} V$ are projective.

**Example**

The lattice of normal subgroups of a $p$-group has the AP property.

**Theorem (E. Aichinger, M. Lazić, 2013)**

Let $V$ be a finite expanded group. $\text{Id} V$ has the AP property iff it is a coalesced ordered sum of simple lattices.
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The lattice of normal subgroups of a $p$-group has the AP property.

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Let $V$ be a finite expanded group. $\text{Id} V$ has the AP property iff it is a coalesced ordered sum of simple lattices.
An induced expanded group

For an ideal $I$ of an expanded group $V$, let us denote $V|_I := (I, \{c|_I : c \in \text{Comp}(V), c(\text{Iar}(c)) \subseteq I\})$.

Restriction on an ideal

Let $V$ be a finite expanded group such that $\text{Comp}(V)$ is generated by the set of functions $\{f_1, \ldots, f_m\}$ and let $I > 0$ be an ideal that splits $\text{Id} V$. Is $\text{Comp}(V|_I)$ also finitely generated?
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A problem with a composition

For an ideal $I$ of an expanded group $V$, it can happen $f \circ g \in \text{Comp}(V|_I)$, but $f(I_{\text{ar}}(f)) \not\subseteq I$ and $g(I_{\text{ar}}(g)) \not\subseteq I$.

Remark

Let $V$ be a finite expanded group such that $\text{Comp}(V)$ is generated by the set of functions $\{f_1, \ldots, f_m\}$ and let $I > 0$ be an ideal that splits $\text{Id} V$. We can not take just $\{f_1|_I, \ldots, f_m|_I\}$ for the generating set of $\text{Comp}(V|_I)$. 
A problem with a composition

For an ideal $I$ of an expanded group $V$, it can happen $f \circ g \in \text{Comp}(V|I)$, but $f(\text{lar}(f)) \not\subseteq I$ and $g(\text{lar}(g)) \not\subseteq I$.

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The T Operator

Definition

Let $V$ be a finite expanded group, let $I \in \text{Id} \ V$ such that $|V/I| = k$, $k \in \mathbb{N}$, and let $0 = s_1, \ldots, s_k$ be a transversal modulo $I$. For every $f \in V^V, n \in \mathbb{N}$, and for every $(\alpha_1, \ldots, \alpha_n) \in \{s_1, \ldots, s_k\}^n$ we define $T^n_{(\alpha_1, \ldots, \alpha_n)}(f) : V^n \rightarrow V$ in the following way:

$$T^n_{(\alpha_1, \ldots, \alpha_n)}(f)(x_1, \ldots, x_n) := f(x_1 + \alpha_1, \ldots, x_n + \alpha_n) - r(f(\alpha_1, \ldots, \alpha_n)),$$

for all $x_1, \ldots, x_n \in V$, where $r$ is a function that maps every element $a \in V$ in the representative of its $I$-class.
The Properties of the T Operator

Let \( V \) be a finite expanded group, let \( l \in \text{Id} V \) such that \( |V/l| = k, k \in \mathbb{N} \), and let \( 0 = s_1, \ldots, s_k \) be a transversal modulo \( l \). For every \( f \in V^{V^n}, n \in \mathbb{N} \), and for every \( (\alpha_1, \ldots, \alpha_n) \in \{s_1, \ldots, s_k\}^n \) we have

\[
T^n_{(\alpha_1, \ldots, \alpha_n)}(f)(l^n) \subseteq l.
\]

If \( f(l^n) \subseteq l \), since \( 0 = r(f(0, \ldots, 0)) \), then \( T^n_{(0, \ldots, 0)}(f) = f \).
Let $V$ be a finite expanded group, let $I \in \text{Id } V$ such that $|V/I| = k$, $k \in \mathbb{N}$, and let $0 = s_1, \ldots, s_k$ be a transversal modulo $I$. For every $f \in V^{V^n}, n \in \mathbb{N}$, and for every $(\alpha_1, \ldots, \alpha_n) \in \{s_1, \ldots, s_k\}^n$ we have

\[ T^n_{(\alpha_1, \ldots, \alpha_n)}(f)(l^n) \subseteq I. \]

If $f(l^n) \subseteq I$, since $0 = r(f(0, \ldots, 0))$, then $T^n_{(0, \ldots, 0)}(f) = f$. 

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The Clone of Compatible Functions of Special Expanded Groups
The Fundamental Operators

For an arbitrary set $A$, on the set $P_A = \{ f : A^n \to A \mid n \in \mathbb{N} \}$ we define operations $\zeta, \tau, \Delta, \circ, \nabla$, such that

1. $(\zeta f)(x_1, x_2, \ldots, x_n) := f(x_2, \ldots, x_n, x_1),$
2. $(\tau f)(x_1, x_2, x_3, \ldots, x_n) := f(x_2, x_1, x_3, \ldots, x_n),$
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4. $(g \circ f)(x_1, \ldots, x_{m+n-1}) := f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}),$
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A Characterization of Clones

Theorem (R. Pöschel, L. Kalužnin)

A set of functions is a clone if and only if it contains $e_1^1$ and it is closed under $\zeta, \tau, \Delta, \circ$ and $\nabla$.

Identity function

$e_1^1$ is the identity function - a constant in $P_A$. 
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4. $T^{m+n-1}_{(\alpha_1,\ldots,\alpha_{m+n-1})}(g \circ f) = T^m_{(\alpha_1,\ldots,\alpha_m)}(g) \circ T^n_{(r(g(\alpha_1,\ldots,\alpha_m)),\alpha_{m+1},\ldots,\alpha_{m+n-1})}(f);$
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**Proposition**

Let $V$ be a finite expanded group, let $I \in \text{Id} V$ such that $|V/I| = k$, $k \in \mathbb{N}$, and let $0 = s_1, \ldots, s_k$ be a transversal modulo $I$. For every $f \in V^V^n$, $g \in V^V^m$, $n, m \in \mathbb{N}$, and for every $\alpha_i \in \{s_1, \ldots, s_k\}$, $i \in \mathbb{N}$, we have:

1. $T^n_{\alpha_1,\alpha_2,\ldots,\alpha_n}(\zeta f) = \zeta(T^n_{\alpha_2,\ldots,\alpha_n,\alpha_1}(f))$;
2. $T^n_{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n}(\tau f) = \tau(T^n_{\alpha_2,\alpha_1,\alpha_3,\ldots,\alpha_n}(f))$;
3. $T^{n-1}_{\alpha_1,\alpha_2\ldots,\alpha_{n-1}}(\Delta f) = \Delta(T^n_{\alpha_1,\alpha_2\ldots,\alpha_{n-1}}(f))$;
4. $T^{m+n-1}_{\alpha_1,\ldots,\alpha_{m+n-1}}(g \circ f) = T^m_{\alpha_1,\ldots,\alpha_m}(g) \circ T^n_{r(g(\alpha_1,\ldots,\alpha_m)),\alpha_{m+1},\ldots,\alpha_{m+n-1}}(f)$;
5. $T^{n+1}_{\alpha_1,\alpha_2\ldots,\alpha_{n+1}}(\nabla f) = \nabla(T^n_{\alpha_2\ldots,\alpha_{n+1}}(f))$. 
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Let $V$ be a finite expanded group, let $I \in \mathrm{Id} V$ such that $|V/I| = k, k \in \mathbb{N}$, and let $0 = s_1, \ldots, s_k$ be a transversal modulo $I$. For every $f \in V^V^n, g \in V^V^m, n, m \in \mathbb{N}$, and for every $\alpha_i \in \{s_1, \ldots, s_k\}, i \in \mathbb{N}$, we have:

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Nebojša Mudrinski

The Clone of Compatible Functions of Special Expanded Groups
Proposition

Let $V$ be a finite expanded group, let $I \in \text{Id } V$ such that $|V/I| = k$, $k \in \mathbb{N}$, and let $0 = s_1, \ldots, s_k$ be a transversal modulo $I$. For every $f \in V^V^n$, $g \in V^V^m$, $n, m \in \mathbb{N}$, and for every $\alpha_i \in \{s_1, \ldots, s_k\}$, $i \in \mathbb{N}$, we have:

1. $\mathcal{T}^n_{(\alpha_1, \alpha_2, \ldots, \alpha_n)}(\zeta f) = \zeta(\mathcal{T}^n_{(\alpha_2, \ldots, \alpha_n, \alpha_1)}(f))$;
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Proposition

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3. $T_{(\alpha_1, \alpha_2, \ldots, \alpha_n-1)}^{n-1}(\Delta f) = \Delta(T_{(\alpha_1, \alpha_1, \alpha_2, \ldots, \alpha_n-1)}^n(f))$;
4. $T_{(\alpha_1, \ldots, \alpha_m, \alpha_{m+n-1})}^{m+n-1}(g \circ f) = T_{(\alpha_1, \ldots, \alpha_m)}^m(g) \circ T_{(\alpha_1, \ldots, \alpha_m)}^n(r(g(\alpha_1, \ldots, \alpha_m)), \alpha_{m+1}, \ldots, \alpha_{m+n-1})(f)$;
5. $T_{(\alpha_1, \alpha_2, \ldots, \alpha_n+1)}^{n+1}(\nabla f) = \nabla(T_{(\alpha_2, \ldots, \alpha_n+1)}^n(f))$. 

Nebojša Mudrinski
The Clone of Compatible Functions of Special Expanded Groups
Theorem

Let \( V \) be a finite expanded group and let \( I > 0 \) be an ideal that splits \( \text{Id} V \). If \( \text{Comp}(V) \) is finitely generated, then \( \text{Comp}(V|_I) \) is also finitely generated.

The generating functions

If \( \text{Comp}(V) \) is generated by \( \{ f_1, \ldots, f_m \} \) then \( \text{Comp}(V|_I) \) is generated by \( \{ g|_I : g \in B \} \) where \( B \) is the following set

\[
\{ T^{ar(f_i)}_{(\alpha_1, \ldots, \alpha_{ar(f_i)})}(f_i) \mid i \in \{1, \ldots, m\}, \\
(\alpha_1, \ldots, \alpha_{ar(f_i)}) \in \{ s_1, \ldots, s_k \}^{ar(f_i)} \}.
\]
The Answer

**Theorem**

Let $V$ be a finite expanded group and let $I > 0$ be an ideal that splits Id $V$. If $\text{Comp}(V)$ is finitely generated, then $\text{Comp}(V|_I)$ is also finitely generated.

**The generating functions**

If $\text{Comp}(V)$ is generated by $\{f_1, \ldots, f_m\}$ then $\text{Comp}(V|_I)$ is generated by $\{g|_I : g \in B\}$ where $B$ is the following set

$$\left\{ T_{(\alpha_1, \ldots, \alpha_{ar(f_i)})}^{ar(f_i)} (f_i) \mid i \in \{1, \ldots, m\}, \right.$$  

$$\left( \alpha_1, \ldots, \alpha_{ar(f_i)} \right) \in \left\{ s_1, \ldots, s_k \right\}^{ar(f_i)} \right\}.$$
Theorem (E. Aichinger, M. Lazić, N. M., 2013)

Let $V$ be a finite expanded group such that $\text{Id } V$ has the AP-property and $0 = S_0 < S_1 < \cdots < S_n = 1$ are all ideals that split $\text{Id } V$. Then TFAE:

1. $\text{Comp}(V)$ is finitely generated;
2. for every $i \in \{1, \ldots, n\}$ we have $S_{i-1} \prec S_i$ or $I[S_{i-1}, S_i]$ does not have a splitting pair.
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The Result

Theorem (E. Aichinger, M. Lazić, N. M., 2013)

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