Quantum quasigroups

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Quasigroups and loops
Quasigroups and loops

Quasigroup: \((Q, \cdot, /, \backslash)\) with
Quasigroups and loops

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\[ y \setminus (y \cdot x) = x = (x \cdot y) / y \quad \text{and} \quad y \cdot (y \setminus x) = x = (x / y) \cdot y. \]
Quasigroups and loops

**Quasigroup**: \((Q, \cdot, /, \backslash)\) with

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\begin{align*}
y \backslash (y \cdot x) &= x = (x \cdot y)/y & \text{and} & & y \cdot (y \backslash x) &= x = (x/y) \cdot y.
\end{align*}
\]

In a magma \((M, \circ)\), with element \(y\), define
Quasigroups and loops

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In a magma $(M, \circ)$, with element $y$, define

**right multiplication** $R_\circ(y): M \to M; x \mapsto x \circ y$ and
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In a magma \((M, \circ)\), with element \(y\), define

**right multiplication** \(R_\circ(y) : M \to M; x \mapsto x \circ y\) and

**left multiplication** \(L_\circ(y) : M \to M; x \mapsto y \circ x\).
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Quasigroup identities say \(L(y) = L.(y)\) and \(R(y) = R.(y)\) bijective.
Quasigroups and loops

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Quasigroup identities say \(L(y) = L.(y)\) and \(R(y) = R.(y)\) bijective.

Loop: Quasigroup \(Q\) with identity element \(e\) satisfying \(x \cdot e = x = e \cdot x\).
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Entropic algebras
Entropic algebras

Algebra \( (A, \Omega) \) is **entropic** if each operation

\[
\omega: A^{\omega \tau} \rightarrow A; (a_1, \ldots, a_{\omega \tau}) \mapsto a_1, \ldots, a_{\omega \tau} \omega
\]

is a homomorphism.
Entropic algebras

Algebra \((A, \Omega)\) is entropic if each operation

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\]

is a homomorphism.

Examples:
Entropic algebras

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is a homomorphism.

**Examples:**

- Modules over a commutative ring;
Entropic algebras

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Examples:

- Modules over a commutative ring;
- Commutative semigroups, e.g., semilattices;
- Barycentric algebras (with convex combinations as operations);
Entropic algebras

Algebra \((A, \Omega)\) is **entropic** if each operation
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is a homomorphism.

**Examples:**
- Modules over a commutative ring;
- Commutative semigroups, e.g., semilattices;
- Barycentric algebras (with convex combinations as operations);
- Sets.
Tensor products
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Variety/category $\mathbf{V}$ of entropic algebras (homomorphisms as morphisms).
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Write $\mathbf{V}(Z, Y; X)$ for the set of bihomomorphisms from $Z \times Y$ to $X$. 
Tensor products

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The tensor product $Z \otimes Y$ features in the adjointness

$V(Z \otimes Y, X) \cong V(Z, V(Y, X)) \cong V(Z, Y; X)$
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Setting $X = Z \otimes Y$, taking $\text{id}_{Z \otimes Y}$ on left, obtain a bihomomorphism

$$\otimes: Z \times Y \to Z \otimes Y; (z, y) \mapsto z \otimes y$$
Tensor products

Variety/category \( V \) of entropic algebras (homomorphisms as morphisms).

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Write \( V(Z, Y; X) \) for the set of **bihomomorphisms** from \( Z \times Y \) to \( X \).

The **tensor product** \( Z \otimes Y \) features in the adjointness
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Setting \( X = Z \otimes Y \), taking \( \text{id}_{Z \otimes Y} \) on left, obtain a bihomomorphism
\[
\otimes : Z \times Y \to Z \otimes Y; (z, y) \mapsto z \otimes y
\]

**Lemma:** \( Z \otimes Y \) is generated by \( \{ z \otimes y \mid z \in Z, y \in Y \} \)
$V$ as a tensor category
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Define isomorphism $\tau: Z \otimes Y \rightarrow Y \otimes Z; z \otimes y \mapsto y \otimes z$. 
\( V \) as a tensor category

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Free algebra \( 1 \) on one generator \( x \) in \( V \).
V as a tensor category

Define isomorphism \( \tau : Z \otimes Y \to Y \otimes Z \); \( z \otimes y \mapsto y \otimes z \).

Free algebra \( \mathbf{1} \) on one generator \( x \) in \( \mathbf{V} \).

Define isomorphisms \( \mathbf{1} \otimes A \xrightarrow{\lambda_A} A \xleftarrow{\rho_A} A \otimes \mathbf{1} \)

by \( x \otimes a \xrightarrow{\lambda_A} a \xleftarrow{\rho_A} a \otimes x \) for \( a \in A \).
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by $x \otimes a \mapsto a \xleftarrow{\rho_A} a \otimes x$ for $a \in A$.

Define isomorphism $\alpha_{C,B,A}: (C \otimes B) \otimes A \to C \otimes (B \otimes A)$

by $(c \otimes b) \otimes a \mapsto c \otimes (b \otimes a)$.
\[ V \text{ as a tensor category} \]

Define isomorphism \( \tau : Z \otimes Y \rightarrow Y \otimes Z; z \otimes y \mapsto y \otimes z. \)

Free algebra \( \mathbf{1} \) on one generator \( x \) in \( V \).

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Define isomorphism \( \alpha_{C,B,A} : (C \otimes B) \otimes A \rightarrow C \otimes (B \otimes A) \)

by \( (c \otimes b) \otimes a \mapsto c \otimes (b \otimes a). \) (Write \( c \otimes b \otimes a \) for identified image.)
\textbf{V as a tensor category}

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\textbf{Proposition:} $\left( \textbf{V}, \otimes, \textbf{1} \right)$ with $\tau$ is a \textbf{symmetric monoidal} (or tensor) \textbf{category}
**V as a tensor category**

Define isomorphism $\tau: Z \otimes Y \rightarrow Y \otimes Z; z \otimes y \mapsto y \otimes z$.

Free algebra $1$ on one generator $x$ in $V$.

Define isomorphisms $1 \otimes A \xrightarrow{\lambda_A} A \xleftarrow{\rho_A} A \otimes 1$
by $x \otimes a \mapsto a \otimes x \text{ for } a \in A$.

Define isomorphism $\alpha_{C,B,A}: (C \otimes B) \otimes A \rightarrow C \otimes (B \otimes A)$
by $(c \otimes b) \otimes a \mapsto c \otimes (b \otimes a)$. (Write $c \otimes b \otimes a$ for identified image.)

**Proposition:** $(V, \otimes, 1)$ with $\tau$ is a **symmetric monoidal** (or tensor) **category**
(“commutative Monoid”).
Symmetric monoidal functors
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Typical examples of symmetric monoidal categories:
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- \((\mathsf{Set}, \times, T)\);
Symmetric monoidal functors

Typical examples of symmetric monoidal categories:

- $(\text{Set}, \times, T)$;
- $(S, \otimes, s)$ for a commutative ring $S$;
Symmetric monoidal functors

Typical examples of symmetric monoidal categories:

- \((\text{Set}, \times, T)\);
- \((S, \otimes, S)\) for a commutative ring \(S\);
- Any entropic variety \((V, \otimes, 1)\);
Symmetric monoidal functors

Typical examples of symmetric monoidal categories:

- \((\text{Set}, \times, \top)\);
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- Any category \((C, +, \bot)\) with coproduct + and initial object \(\bot\).
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A symmetric monoidal functor is a Monoid homomorphism.
Symmetric monoidal functors

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**Examples:**
Symmetric monoidal functors

Typical examples of symmetric monoidal categories:

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A symmetric monoidal functor is a Monoid homomorphism.

Examples:

- Free algebra functor \(F : (\text{Set}, \times, \top) \to (V, \otimes, 1)\) for an entropic variety \(V\);
Symmetric monoidal functors

Typical examples of symmetric monoidal categories:

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A symmetric monoidal functor is a Monoid homomorphism.

Examples:

- Free algebra functor \(F : (\text{Set}, \times, \top) \to (V, \otimes, 1)\) for an entropic variety \(V\);
- Underlying set functor \(U : (S, \oplus, \{0\}) \to (\text{Set}, \times, \top)\).
Monoid and comonoid diagrams
Monoid and comonoid diagrams

\[ A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \]

\[ \nabla \otimes 1_A \]

\[ A \otimes A \xrightarrow{\nabla} A \]

\[ 1 \otimes A \xrightarrow{\lambda_A} A \]

\[ \text{monoid} \]

\[ A \otimes A \xleftarrow{1_A \otimes \eta} A \otimes 1 \]

\[ \eta \otimes 1_A \]

\[ A \otimes 1 \xrightarrow{\rho_A} A \]
Monoid and comonoid diagrams

\[
A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \\
\nabla \otimes 1_A \\
A \otimes A \xrightarrow{\nabla} A
\]

unit \(\eta\), multiplication \(\nabla\)

\[
A \otimes A \xleftarrow{\eta \otimes 1_A} A \otimes 1 \\
A \otimes A \xrightarrow{\eta \otimes 1_A} \nabla \\
\rho_A \nabla \downarrow \\
1 \otimes A \xrightarrow{\lambda_A} A
\]
Monoid and comonoid diagrams

\[
\begin{align*}
\text{monoid} & \quad A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \\
& \quad \eta \otimes 1_A \quad \nabla \quad \rho_A \\
& \quad 1 \otimes A \xleftarrow{\lambda_A} A
\end{align*}
\]

unit \( \eta \), multiplication \( \nabla \)

\[
\begin{align*}
\text{comonoid} & \quad A \otimes A \otimes A \xleftarrow{1_A \otimes \Delta} A \otimes A \\
& \quad \varepsilon \otimes 1_A \quad \Delta \quad \rho_A^{-1} \\
& \quad 1 \otimes A \xleftarrow{\lambda_A^{-1}} A
\end{align*}
\]
Monoid and comonoid diagrams

unit $\eta$, multiplication $\nabla$

counit $\varepsilon$, comultiplication in Sweedler notation $\Delta: a \mapsto a^L \otimes a^R$ or $\Delta: A \rightarrow A \otimes A; a \mapsto (a^{L_1} \otimes a^{R_1}) \ldots (a^{L_{n_a}} \otimes a^{R_{n_a}})w_a$
Bi-algebra diagram
Bi-algebra diagram

\[ \begin{array}{ccc}
1 \otimes 1 & \xrightarrow{\nabla} & 1 \\
A \otimes A & \xrightarrow{\nabla} & A \\
A \otimes A \otimes A \otimes A & \xrightarrow{1_A \otimes \tau \otimes 1_A} & A \otimes A \otimes A \otimes A \\
\end{array} \]
Bi-algebra diagram

\[
\begin{align*}
\Delta &\quad \eta \\
\n\varepsilon \otimes \varepsilon &\quad \epsilon \\
\Delta \otimes \Delta &\quad \nabla \otimes \nabla \\
1 \otimes 1 &\quad 1 \otimes 1 \\
A \otimes A &\quad A \otimes A \\
A \otimes A \otimes A \otimes A &\quad A \otimes A \otimes A \otimes A
\end{align*}
\]

means \(\{\Delta, \nabla\}\) is a \(\{\text{monoid, comonoid}\}\) homomorphism.
Antipode diagram
Antipode diagram
Bi-algebra with an antipode $S$ is a Hopf algebra or quantum group.
Examples of Hopf algebras
Examples of Hopf algebras

- In \((\textbf{Set}, \times, \top)\), a group \((G, \cdot, 1)\) with:

\[
\begin{align*}
\nabla &: g \otimes h \mapsto gh; \\
\Delta &: g \mapsto g \otimes g; \\
\eta &: \top \mapsto \{1\}; \\
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- Applying the free algebra functor \(F : \text{Set} \to \mathbf{V}\) yields a group algebra.
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- In \(\mathbb{S}\), dualizing (for \(G\) finite) yields a dual group algebra.
Examples of Hopf algebras

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• Applying the free algebra functor $F : \text{Set} \to \mathbf{V}$ yields a group algebra.

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• Combining these constructions gives the quantum double of a finite group.
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• In \(S\), the universal enveloping algebra \(U(L)\) of a Lie algebra \(L\).
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- In \(S\), the universal enveloping algebra \(U(L)\) of a Lie algebra \(L\),
  with \(\Delta : \ x \mapsto x \otimes 1 + 1 \otimes x\) for \(x \in L\),
Examples of Hopf algebras

- In \((\text{Set}, \times, \top)\), a group \((G, \cdot, 1)\) with:
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  \nabla: & \quad g \otimes h \mapsto gh; \\
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- In \(\mathbb{S}\), the universal enveloping algebra \(U(L)\) of a Lie algebra \(L\),
  with \(\Delta : x \mapsto x \otimes 1 + 1 \otimes x\) for \(x \in L\),
  and \(\nabla\) as the linearized algebra multiplication.
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QUANTUM QUASIGROUPS

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Magmas and comagmas
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Magma \((A, \nabla: A \otimes A \to A)\)
Magmas and comagmas

Magma \((A, \nabla: A \otimes A \to A)\)

Unital magma \((A, \nabla: A \otimes A \to A, \eta: 1 \to A)\)
Magmas and comagmas

Magma \((A, \triangledown : A \otimes A \to A)\)

Unital magma \((A, \triangledown : A \otimes A \to A, \eta : 1 \to A)\) with

\[
\begin{array}{ccc}
A \otimes A & \xleftarrow{1_A \otimes \eta} & A \otimes 1 \\
\eta \otimes 1_A & \downarrow \triangledown & \downarrow \rho_A \\
1 \otimes A & \xrightarrow{\lambda_A} & A
\end{array}
\]
Magmas and comagmas

**Magma** \((A, \nabla : A \otimes A \to A)\)

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\[
\begin{array}{c}
A \otimes A \\
\eta \otimes 1_A \\
1 \otimes A
\end{array}
\xleftarrow{\nabla} \xrightarrow{1_A \otimes \eta} \xrightarrow{\lambda_A} \xrightarrow{} A
\]

**Comagma** \((A, \Delta : A \to A \otimes A)\)
Magmas and comagmas

Magma \((A, \nabla: A \otimes A \to A)\)

Unital magma \((A, \nabla: A \otimes A \to A, \eta: 1 \to A)\) with

\[
\begin{array}{ccc}
A \otimes A & \overset{1_A \otimes \eta}{\longrightarrow} & A \otimes 1 \\
\eta \otimes 1_A & \downarrow & \nabla \\
1 \otimes A & \overset{\lambda_A}{\longrightarrow} & A \\
\end{array}
\]

Comagma \((A, \Delta: A \to A \otimes A)\)

Counital magma \((A, \Delta: A \to A \otimes A, \varepsilon: A \to 1)\)
Magmas and comagmas

Magma \((A, \nabla : A \otimes A \to A)\)

Unital magma \((A, \nabla : A \otimes A \to A, \eta : 1 \to A)\) with \(\eta \otimes 1_A\)

Comagma \((A, \Delta : A \to A \otimes A)\)

Counital magma \((A, \Delta : A \to A \otimes A, \varepsilon : A \to 1)\) with \(\varepsilon \otimes 1_A\)
Comagmas in \((\text{Set}, \times, T)\)
Comagmas in \((\mathbb{S}et, \times, \top)\)

General comagma \((A, \Delta)\) in \((\mathbb{S}et, \times, \top)\) is \(\Delta: A \rightarrow A \otimes A; a \mapsto a^L \otimes a^R\)
Comagmas in \((\text{Set}, \times, \top)\)

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with functions \(L : A \rightarrow A; a \mapsto a^L\) and \(R : A \rightarrow A; a \mapsto a^R\).
Comagmas in \((\mathbf{Set}, \times, \top)\)

General comagma \((A, \Delta)\) in \((\mathbf{Set}, \times, \top)\) is \(\Delta : A \to A \otimes A; a \mapsto a^L \otimes a^R\)

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**Proposition:** If \((A, \Delta)\) is counital, then \(\Delta : a \mapsto a \otimes a\).
**Comagmas in** \((\mathbf{Set}, \times, \top)\)

General comagma \((A, \Delta)\) in \((\mathbf{Set}, \times, \top)\) is \(\Delta : A \to A \otimes A; a \mapsto a^L \otimes a^R\)

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**Proposition:** If \((A, \Delta)\) is counital, then \(\Delta : a \mapsto a \otimes a\). (Each element \(a\) is setlike.)
Comagmas in \((\text{Set}, \times, \top)\)

General comagma \((A, \Delta)\) in \((\text{Set}, \times, \top)\) is \(\Delta : A \rightarrow A \otimes A; a \mapsto a^L \otimes a^R\)

with functions \(L : A \rightarrow A; a \mapsto a^L\) and \(R : A \rightarrow A; a \mapsto a^R\).

**Proposition:** If \((A, \Delta)\) is counital, then \(\Delta : a \mapsto a \otimes a\). (Each element \(a\) is setlike.)

**Proof:** The counital diagram

\[
\begin{array}{c}
A \otimes A \xrightarrow{1 \otimes \varepsilon} A \otimes 1 \\
\varepsilon \otimes 1_A \downarrow \quad \Delta \downarrow \quad \rho_A^{-1} \\
1 \otimes A \quad A
\end{array}
\]

\[
\lambda_A^{-1}
\]
Comagmas in \((\text{Set}, \times, \top)\)

General comagma \((A, \Delta)\) in \((\text{Set}, \times, \top)\) is \(\Delta: A \to A \otimes A; a \mapsto a^L \otimes a^R\)
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\[
\begin{array}{c}
\varepsilon \otimes 1_A \downarrow & \Delta \downarrow & \rho_A^{-1} \\
A \otimes A & A \otimes 1 & \\
1 \otimes A & A \\
\hline
1 \otimes A & A
\end{array}
\]

yields \(a^L \otimes a^R \mapsto a \otimes x\),

\[
\begin{array}{c}
\varepsilon \otimes 1_A \downarrow & \Delta \downarrow & \rho_A^{-1} \\
x \otimes a & a \\
\hline
x \otimes a & a
\end{array}
\]
**Comagmas in** $\langle \text{Set}, \times, \top \rangle$

General comagma $(A, \Delta)$ in $\langle \text{Set}, \times, \top \rangle$ is $\Delta : A \to A \otimes A; a \mapsto a^L \otimes a^R$

with functions $L : A \to A; a \mapsto a^L$ and $R : A \to A; a \mapsto a^R$.

**Proposition:** If $(A, \Delta)$ is counital, then $\Delta : a \mapsto a \otimes a$. (Each element $a$ is *setlike*.)

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A \otimes A & \xrightarrow{1_A \otimes \varepsilon} & A \otimes 1 \\
\downarrow \varepsilon \otimes 1_A & & \Delta \\
1 \otimes A & \xleftarrow{\rho^{-1}_A} & A
\end{array}
\]

yields $\Delta : a \mapsto a \otimes a$, so $a^L = a = a^R$. $\Box$
Bimagmas
Bimagmas

Bimagma \((A, \nabla, \Delta)\) with
**Bimagmas**

**Bimagma** \((A, \nabla, \Delta)\) with

\[
\begin{array}{c}
\Delta \\
\nabla \\
A \otimes A
\end{array}
\quad
\begin{array}{c}
A \\
\Delta \\
A \otimes A
\end{array}
\]

\[
\begin{array}{c}
\Delta \otimes \Delta \\
A \otimes A \otimes A \otimes A
\end{array}
\quad
\begin{array}{c}
\nabla \otimes \nabla \\
A \otimes A \otimes A \otimes A
\end{array}
\]

\[
1_A \otimes \tau \otimes 1_A
\]

So \(\Delta \) is a \(\nabla\) magma \(\Delta\) comagma homomorphism.
Biunital bimagmas
Biunital bimagmas

A biunital bimagma is a unital and counital bimagma \((A, \nabla, \Delta, \eta, \varepsilon)\)
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So \(\left\{ \begin{array}{c} \Delta \\ \nabla \end{array} \right\} \) is a unital magma \(\) unital magma \(\) counital comagma \(\) counital comagma \(\) homomorphism.
Quantum quasigroups and loops
Quantum quasigroups and loops

A quantum quasigroup is a bimagma \((A, \nabla, \Delta)\) with invertible
Quantum quasigroups and loops

A quantum quasigroup is a bimagma \((A, \nabla, \Delta)\) with invertible left composite

\[
\begin{align*}
A \otimes A &\xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \\
\end{align*}
\]
Quantum quasigroups and loops

A quantum quasigroup is a bimagma \((A, \nabla, \Delta)\) with invertible

left composite \[ A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \] and

right composite \[ A \otimes A \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\nabla \otimes 1_A} A \otimes A \].
Quantum quasigroups and loops

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A quantum loop is a biunital bimagma \((A, \nabla, \Delta, \eta, \varepsilon)\)
in which the reduct \((A, \nabla, \Delta)\) is a quantum quasigroup.
Quantum quasigroups and loops

A quantum quasigroup is a bimagma \((A, \nabla, \Delta)\) with invertible left composite

\[ A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \]

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Remark: These definitions are self-dual,
Quantum quasigroups and loops

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A quantum loop is a biunital bimagma \((A, \nabla, \Delta, \eta, \varepsilon)\)
in which the reduct \((A, \nabla, \Delta)\) is a quantum quasigroup.

Remark: These definitions are self-dual,
and invariant under symmetric monoidal functors.
Combinatorial quantum loops
Combinatorial quantum loops

**Proposition:** Counital quantum quasigroups in $(\text{Set}, \times, \top)$ are equivalent to quasigroups.
Combinatorial quantum loops

Proposition: Counital quantum quasigroups in \((\text{Set}, \times, \top)\) are equivalent to quasigroups.

Proof: Left composite is \[ a \otimes b \xrightarrow{\Delta \otimes 1_A} a \otimes a \otimes b \xrightarrow{1_A \otimes \triangledown} a \otimes (a \cdot b) \, . \]
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\]

Inverse is

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  c \otimes (c \backslash d) & \xleftarrow{} c \otimes d.
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for binary operation \(c \backslash d\).
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\Delta \otimes 1_A & \rightarrow c \otimes (c \backslash d) \rightarrow c \otimes c \otimes (c \backslash d) & \rightarrow 1_A \otimes \nabla & \rightarrow c \otimes (c \cdot (c \backslash d))
\end{align*}
\]
Inverse is
\[
\begin{align*}
\quad & c \otimes (c \backslash d) \leftarrow c \otimes d
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Dually, have $(b \cdot a) / a = b$ and $(d / c) \cdot c = d$, so a quasigroup.
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\begin{array}{c}
\text{a} \otimes \text{b} \xrightarrow{\Delta \otimes 1_A} \text{a} \otimes \text{a} \otimes \text{b} \xrightarrow{1_A \otimes \triangledown} \text{a} \otimes (\text{a} \cdot \text{b})
\end{array}
\]
Inverse is
\[
\begin{array}{c}
\text{c} \otimes (\text{c} \backslash \text{d}) \xleftarrow{\quad} \text{c} \otimes \text{d}
\end{array}
\]
for binary operation \(\text{c} \backslash \text{d}\) with \(\text{b} = \text{a} \backslash (\text{a} \cdot \text{b})\) and \(\text{d} = \text{c} \cdot (\text{c} \backslash \text{d})\).

Dually, have \((\text{b} \cdot \text{a}) / \text{a} = \text{b}\) and \((\text{d} / \text{c}) \cdot \text{c} = \text{d}\), so a quasigroup.

Conversely, a quasigroup \((Q, \cdot, /, \backslash)\) provides a counital quantum quasigroup. \(\square\)
Combinatorial quantum loops

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\end{align*}
\]

Inverse is

\[
\begin{align*}
& c \otimes (c \downarrow d) \leftarrow c \otimes d
\end{align*}
\]

for binary operation \(c \downarrow d\) with \(b = a \downarrow (a \cdot b)\) and \(d = c \cdot (c \downarrow d)\).

Dually, have \((b \cdot a)/a = b\) and \((d/c) \cdot c = d\), so a quasigroup.

Conversely, a quasigroup \((Q, \cdot, /, \backslash)\) provides a counital quantum quasigroup. \(\square\)

**Corollary:** Quantum loops in \((\text{Set}, \times, \top)\) are equivalent to loops.
Combinatorial quantum quasigroups
Combinatorial quantum quasigroups

Proposition: Finite quantum quasigroups in \((\textbf{Set}, \times, T)\) are equivalent to quasigroups with an ordered pair \((L, R)\) of automorphisms.
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a \otimes b \xrightarrow{\Delta \otimes 1_A} a^L \otimes a^R \otimes b \xrightarrow{1_A \otimes \triangledown} a^L \otimes (a^R \cdot b)
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Invertibility implies \(L\) surjective; dually, \(R\).
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Derive quasigroup identities.
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Derive quasigroup identities. Converse clear. \(\square\)
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Derive quasigroup identities. Converse clear. \(\square\)

**Problem:** Classify arbitrary quantum quasigroups in \((\text{Set}, \times, \top)\).
The quantum couple
The quantum couple

**Theorem:** Group $G$ with automorphic right action on finite quasigroup $Q$. 
The quantum couple

Theorem: Group $G$ with automorphic right action on finite quasigroup $Q$. Commutative ring $S$, tensor product $GQ$ of free modules $GS$ and $QS$. 
The quantum couple

**Theorem:** Group $G$ with automorphic right action on finite quasigroup $Q$.
Commutative ring $S$, tensor product $GQ$ of free modules $GS$ and $QS$.
For $g \in G$ and $q \in Q$, write $g|q$ for $g \otimes q$. 
The quantum couple

**Theorem:** Group $G$ with automorphic right action on finite quasigroup $Q$. Commutative ring $S$, tensor product $GQ$ of free modules $GS$ and $QS$.

For $g \in G$ and $q \in Q$, write $g\mid q$ for $g \otimes q$.

Define $(f|p \otimes g|q)\nabla = \begin{cases} fg|q & \text{if } pg = q; \\ 0 & \text{otherwise} \end{cases}$
The quantum couple

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and $\Delta : g|q \mapsto \sum_{q^L q^R = q} g|q^L \otimes g|q^R$. Extend these maps by linearity.
The quantum couple

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and $\Delta : g|q \mapsto \sum_{qLqR=q} g|qL \otimes g|qR$. Extend these maps by linearity.

Then the **quantum couple** $(GQ, \nabla, \Delta)$ is an associative quantum quasigroup in $S$. 
The quantum couple

**Theorem:** Group $G$ with automorphic right action on finite quasigroup $Q$.
Commutative ring $S$, tensor product $GQ$ of free modules $GS$ and $QS$.

For $g \in G$ and $q \in Q$, write $g\mid q$ for $g \otimes q$.

Define $(f\mid p \otimes g\mid q) \nabla = \begin{cases} fg\mid q & \text{if } pg = q; \\ 0 & \text{otherwise} \end{cases}$

and $\Delta: g\mid q \mapsto \sum_{qLq^R=q} g\mid q^L \otimes g\mid q^R$. Extend these maps by linearity.

Then the **quantum couple** $(GQ, \nabla, \Delta)$ is an associative quantum quasigroup in $S$.

- If $G = \top$, then $\top Q$ is a **dual quasigroup algebra**.
The quantum couple

**Theorem:** Group $G$ with automorphic right action on finite quasigroup $Q$.

Commutative ring $S$, tensor product $GQ$ of free modules $GS$ and $QS$.

For $g \in G$ and $q \in Q$, write $g|q$ for $g \otimes q$.

Define $\left( f|p \otimes g|q \right) \nabla = \begin{cases} fg|q & \text{if } pg = q; \\ 0 & \text{otherwise} \end{cases}$

and $\Delta : g|q \mapsto \sum q_L q_R = q g|q^L \otimes g|q^R$. Extend these maps by linearity.

Then the **quantum couple** $(GQ, \nabla, \Delta)$ is an associative quantum quasigroup in $S$.

- If $G = \top$, then $\top Q$ is a **dual quasigroup algebra**.
- If $Q = \top$, then $G\top$ is the (quasi-)group algebra $GS$ of $G$. 
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**Theorem:** Group $G$ with automorphic right action on finite quasigroup $Q$.

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For $g \in G$ and $q \in Q$, write $g|q$ for $g \otimes q$.

Define 

$$(f|p \otimes g|q)\nabla = \begin{cases} fg|q & \text{if } pg = q; \\ 0 & \text{otherwise} \end{cases}$$

and $\Delta : g|q \mapsto \sum_{qL qR = q} g|qL \otimes g|qR$. Extend these maps by linearity.

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- If $G = \top$, then $\top Q$ is a **dual quasigroup algebra**.
- If $Q = \top$, then $G\top$ is the (quasi-)group algebra $GS$ of $G$.
- If finite $G$ acts on $G$ by conjugation, then $GG$ is the group quantum double.
Hopf algebras as quantum loops
Hopf algebras as quantum loops

**Theorem:** If $(A, \nabla, \Delta, \eta, \varepsilon, S)$ is a Hopf algebra, the reduct $(A, \nabla, \Delta, \eta, \varepsilon)$ is a quantum loop.
Hopf algebras as quantum loops

Theorem: If $(A, \nabla, \Delta, \eta, \varepsilon, S)$ is a Hopf algebra,
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Proof: Coassociativity gives $x^{LL} \otimes x^{LR} \otimes x^R = x^L \otimes x^{RL} \otimes x^{RR}$. 
Hopf algebras as quantum loops

**Theorem:** If \((A, \nabla, \Delta, \eta, \varepsilon, S)\) is a Hopf algebra, the reduct \((A, \nabla, \Delta, \eta, \varepsilon)\) is a quantum loop.

**Proof:** Coassociativity gives \(x^{LL} \otimes x^{LR} \otimes x^R = x^L \otimes x^{RL} \otimes x^{RR}\).

Applying \(\otimes y\) and \((1_A \otimes 1_A \otimes S \otimes 1_A)(1_A \otimes 1_A \otimes \nabla)(1_A \otimes \nabla)\) gives

\[x^{LL} \otimes x^{LR} x^{RS} y = x^L \otimes x^{RL} x^{RRS} y\]
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\[x^{LL} \otimes x^{LR} \otimes x^R = x^L \otimes x^{RL} \otimes x^{RR}.\]
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\[x^{LL} \otimes x^{LR} x^{RS} y = x^L \otimes x^{RL} x^{RRS} y = x^L \otimes x^{R \varepsilon \eta} y\]
Hopf algebras as quantum loops

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x^{LL} \otimes x^{LR} \otimes x^{R} = x^{L} \otimes x^{RL} \otimes x^{RR}.
\]

Applying \( \otimes y \) and \( (1_{A} \otimes 1_{A} \otimes S \otimes 1_{A})(1_{A} \otimes 1_{A} \otimes ∇)(1_{A} \otimes ∇) \) gives

\[
x^{LL} \otimes x^{LR} x^{RS} y = x^{L} \otimes x^{RL} x^{RRS} y
\]

\[
= x^{L} \otimes x^{Rεη} y \quad \text{[by the antipode property } a^{L} a^{RS} = a^{εη}]\]
Hopf algebras as quantum loops

**Theorem:** If \((A, \nabla, \Delta, \eta, \varepsilon, S)\) is a Hopf algebra,

the reduct \((A, \nabla, \Delta, \eta, \varepsilon)\) is a quantum loop.

**Proof:** Coassociativity gives

\[
x^{LL} \otimes x^{LR} \otimes x^R = x^L \otimes x^{RL} \otimes x^{RR}.
\]

Applying \(\otimes y\) and \((1_A \otimes 1_A \otimes S \otimes 1_A)(1_A \otimes 1_A \otimes \nabla)(1_A \otimes \nabla)\) gives

\[
x^{LL} \otimes x^{LR} x^{RS} y = x^L \otimes x^{RL} x^{RRS} y
\]

\[
= x^L \otimes x^{R\varepsilon} y \quad \text{[by the antipode property } a^L a^{RS} = a^{\varepsilon\eta}] \\
= x^L x^{R\varepsilon} \otimes y
\]
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**Theorem:** If \((A, \nabla, \Delta, \eta, \varepsilon, S)\) is a Hopf algebra,

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**Proof:** Coassociativity gives \(x_{LL} \otimes x_{LR} \otimes x_R = x_L \otimes x_{RL} \otimes x_{RR} \).

Applying \(\otimes y\) and \((1_A \otimes 1_A \otimes S \otimes 1_A)(1_A \otimes 1_A \otimes \nabla)(1_A \otimes \nabla)\) gives

\[x_{LL} \otimes x_{LR} x_{RS} y = x_L \otimes x_{RL} x_{RRS} y\]

\[= x_L \otimes x_{R\varepsilon \eta} y \quad \text{[by the antipode property } a_L a_{RS} = a^{\varepsilon \eta}]\]

\[= x_L x_{R\varepsilon \eta} \otimes y \quad \text{[moving “scalars” round the tensor product]}\]
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\[x^{LL} \otimes x^{LR} x^{RS} y = x^L \otimes x^{RL} x^{RRS} y\]
\[= x^L \otimes x^{R\varepsilon \eta} y \quad \text{[by the antipode property} a^L a^{RS} = a^{\varepsilon \eta}]\]
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Proof: Coassociativity gives

\[ x^{LL} \otimes x^{LR} \otimes x^R = x^L \otimes x^{RL} \otimes x^{RR}. \]

Applying \(\otimes y\) and \((1_A \otimes 1_A \otimes S \otimes 1_A)(1_A \otimes 1_A \otimes \nabla)(1_A \otimes \nabla)\) gives

\[ x^{LL} \otimes x^{LR} x^{RS} y = x^L \otimes x^{RL} x^{RRS} y \]

\[ = x^L \otimes x^{R\varepsilon y} \quad [\text{by the antipode property } a^L a^{RS} = a^{\varepsilon y}] \]

\[ = x^L x^{R\varepsilon} \otimes y \quad [\text{moving “scalars” round the tensor product}] \]

\[ = x \otimes y \quad [\text{by the counitality property } a^L a^{R\varepsilon y} = a] \]
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\[x^{LL} \otimes x^{LR} x^{RS} y = x^L \otimes x^{RL} x^{RRS} y\]

\[= x^L \otimes x^{R \varepsilon \eta} y \quad \text{[by the antipode property} \quad a^L a^{RS} = a^{\varepsilon \eta}]\]

\[= x^L x^{R \varepsilon \eta} \otimes y \quad \text{[moving “scalars” round the tensor product]}\]

\[= x \otimes y \quad \text{[by the counitality property} \quad a^L a^{R \varepsilon \eta} = a]\]

so \((\Delta \otimes 1_A)(1_A \otimes S \otimes 1_A)(1_A \otimes \nabla))((\Delta \otimes 1_A)(1_A \otimes \nabla)) = 1_A \otimes A\).
Hopf algebras as quantum loops

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Proof: Coassociativity gives \(x^{LL} \otimes x^{LR} \otimes x^R = x^L \otimes x^{RL} \otimes x^{RR}\).

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x^{LL} \otimes x^{LR} x^{RS} y = x^L \otimes x^{RL} x^{RRS} y
= x^L \otimes x^{R\varepsilon\eta} y \quad \text{[by the antipode property \(a^L a^{RS} = a^{\varepsilon\eta}\)]}
= x^L x^{R\varepsilon\eta} \otimes y \quad \text{[moving “scalars” round the tensor product]}
= x \otimes y \quad \text{[by the counitality property \(a^L a^{R\varepsilon\eta} = a\)]}
\]

so \(\left((\Delta \otimes 1_A)(1_A \otimes S \otimes 1_A)(1_A \otimes \nabla)\right)\left((\Delta \otimes 1_A)(1_A \otimes \nabla)\right) = 1_A \otimes 1_A\).

Other invertibility verifications similar. \(\square\)
Pointed comonoids
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Category $\mathcal{K}$ of vector spaces over field $K$. 
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A comonoid is **simple** if it has exactly two subcomonoids

(one trivial, the other improper).
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A comonoid is **simple** if it has exactly two subcomonoids

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Bimonoid is **pointed** if its comonoid reduct is.
Quantum loops as Hopf algebras
Quantum loops as Hopf algebras

**Theorem:** If \((A, \nabla, \Delta, \eta, \varepsilon)\) is an associative, coassociative finite-dimensional quantum loop in \(K\), and \((A, \Delta, \varepsilon)\) is pointed, then \((A, \nabla, \Delta, \eta, \varepsilon)\) is a Hopf algebra.
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**Proof:** Finite set \(A_1 = \{x \in A \mid a\Delta = a \otimes a, \ a\varepsilon = 1\}\) of setlike elements of \(A\) forms a monoid under multiplication, identity \(1\eta = e\).
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Proof: Finite set \(A_1 = \{x \in A \mid a\Delta = a \otimes a, a\varepsilon = 1\} \) of setlike elements of \(A\) forms a monoid under multiplication, identity \(1\eta = e\). For \(x, y \in A_1\), injective left composite
\[
x \otimes y \xrightarrow{\Delta \otimes 1_A} x \otimes x \otimes y \xrightarrow{1_A \otimes \nabla} x \otimes xy
\] gives bijective \(j : A_1 \otimes A_1 \to A_1 \otimes A_1\).
Quantum loops as Hopf algebras

**Theorem:** If \((A, \nabla, \Delta, \eta, \varepsilon)\) is an associative, coassociative finite-dimensional quantum loop in \(K\), and \((A, \Delta, \varepsilon)\) is pointed, then \((A, \nabla, \Delta, \eta, \varepsilon)\) is a Hopf algebra.

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Quantum loops as Hopf algebras

Theorem: If \((A, \nabla, \Delta, \eta, \varepsilon)\) is an associative, coassociative finite-dimensional quantum loop in \(K\), and \((A, \Delta, \varepsilon)\) is pointed, then \((A, \nabla, \Delta, \eta, \varepsilon)\) is a Hopf algebra.

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\[
\forall u \in A_1, \exists v, w \in A_1. w \otimes wv = (w \otimes v)j = u \otimes e, \text{ whence } w = u \text{ and } uv = e.
\]
Dually, \(\exists v' \in A_1. v'u = e\):
Quantum loops as Hopf algebras

Theorem: If $(A, \nabla, \Delta, \eta, \varepsilon)$ is an associative, coassociative finite-dimensional quantum loop in $K$, and $(A, \Delta, \varepsilon)$ is pointed, then $(A, \nabla, \Delta, \eta, \varepsilon)$ is a Hopf algebra.

Proof: Finite set $A_1 = \{x \in A \mid a\Delta = a \otimes a, \ a\varepsilon = 1\}$ of setlike elements of $A$ forms a monoid under multiplication, identity $1\eta = e$. For $x, y \in A_1$, injective left composite $x \otimes y \xrightarrow{\Delta \otimes 1_A} x \otimes x \otimes y \xrightarrow{1_A \otimes \nabla} x \otimes xy$ gives bijective $j: A_1 \otimes A_1 \to A_1 \otimes A_1$. So $\forall u \in A_1, \exists v, w \in A_1. w \otimes wv = (w \otimes v)j = u \otimes e$, whence $w = u$ and $uv = e$.

Dually, $\exists v' \in A_1. v'u = e$: Each setlike element is invertible.
**Quantum loops as Hopf algebras**

**Theorem:** If \((A, \nabla, \Delta, \eta, \varepsilon)\) is an associative, coassociative finite-dimensional quantum loop in \(K\), and \((A, \Delta, \varepsilon)\) is pointed, then \((A, \nabla, \Delta, \eta, \varepsilon)\) is a Hopf algebra.

**Proof:** Finite set \(A_1 = \{x \in A \mid a\Delta = a \otimes a, \ a\varepsilon = 1\}\) of setlike elements of \(A\) forms a monoid under multiplication, identity \(1\eta = e\). For \(x, y \in A_1\), injective left composite
\[
x \otimes y \xrightarrow{\Delta \otimes 1_A} x \otimes x \otimes y \xrightarrow{1_A \otimes \nabla} x \otimes xy
\]
gives bijective \(j : A_1 \otimes A_1 \to A_1 \otimes A_1\). So \(\forall u \in A_1, \ \exists v, w \in A_1, w \otimes wv = (w \otimes v)j = u \otimes e, \) whence \(w = u\) and \(uv = e\).

Dually, \(\exists v' \in A_1, v'u = e\): Each setlike element is invertible.

Since the bimonoid \((A, \nabla, \Delta, \eta, \varepsilon)\) is pointed, Radford [Proposition 7.6.3] gives a Hopf algebra \((A, \nabla, \Delta, \eta, \varepsilon, S)\). \(\square\)
Thank you for your attention!