Weighted clones

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Joint work with Libor Barto

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Find $x_1, x_2, x_3, x_4, x_5 \in D$ such that

$$R_1(x_1, x_2, x_3) \land R_2(x_2, x_4) \land R_3(x_3, x_5); \quad R_1, R_2, R_3 \in \text{Rel}_D$$

Polymorphism $f \in Op_D$ preserves solutions of CSP:

$$R_2(x_{11}, x_{12}, x_{13}) \Rightarrow f(x_{11}, x_{12}, x_{13}) \land f(x_{21}, x_{22}, x_{23})$$

$R_2$ is invariant under $f$. 
Find $x_1, x_2, x_3, x_4, x_5 \in D$ such that

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Polymorphism $f \in Op_D$ preserves solutions of CSP:

$R_2$ is invariant under $f$. 

$R_2$ is $\forall x_1, x_2, x_3 \in D$ such that $R_1(x_1, x_2, x_3) \land R_2(x_2, x_4) \land R_3(x_3, x_5) ; R_1, R_2, R_3 \in \text{Rel}_D$.
Galois correspondence
Clones and relational clones

\[ \mathcal{F} \subseteq \text{Op}_D \]

**Clone(\mathcal{F}):**
- contains projections \( \pi_i^n \)
- closed under superposition
  - ternary \( f \), binary \( g_1, g_2, g_3 \)
  - \[ f[g_1, g_2, g_3](x_1, x_2) = f(g_1(x_1, x_2), g_2(x_1, x_2), g_3(x_1, x_2)) \]

\[ \mathcal{R} \subseteq \text{Rel}_D \]

**RelClone(\mathcal{R}):**
- contains equality relation
- closed under PP-definition
  - \[ S(x_1, x_2, x_3) := \exists x_4 R_1(x_1, x_2, x_3) \wedge R_2(x_2, x_4) \]
Clones and relational clones

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- contains projections \(\pi^n_i\)
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ternary \(f\), binary \(g_1, g_2, g_3\)

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\]
Galois correspondence

\[ \text{Pol}(\text{Inv}(F)) = \text{Clone}(F) \quad \text{Inv}(\text{Pol}(R)) = \text{RelClone}(R) \]
Valued constraint satisfaction problem

From relations to weighted relations:

- Find $x_1, x_2, x_3, x_4, x_5 \in D$ that minimize

$$R_1(x_1, x_2, x_3) + R_2(x_2, x_4) + R_3(x_3, x_5)$$

- Weighted relations $R_1, R_2, R_3 \in \text{wRel}_D$
- Weighted $r$-ary relation $R$ naturally induces classic $r$-ary relation $R^c$:

$$R^c(x_1, x_2, ..., x_r) \iff R(x_1, x_2, ..., x_r) < +\infty$$

Therefore the polymorphism of $R$ is well defined for a weighted relation $R$. 
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  $$R_1 : D^3 \rightarrow \mathbb{Q} \cup \{+\infty\}, \ R_2, R_3 : D^2 \rightarrow \mathbb{Q} \cup \{+\infty\}$$

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R^c(x_1, x_2, \ldots, x_r) \iff R(x_1, x_2, \ldots, x_r) < +\infty
\]

- Therefore the polymorphism of \( R \) is well defined for a weighted relation \( R \).
Weightings

From clones of operations to weightings:
Let $C \subseteq \text{Op}_D$ be a clone, $C_k$ denotes $k$-ary operations in $C$. Mapping $\omega : C_k \rightarrow \mathbb{Q}$ is a **weighting** if

1. $\omega(f) < 0 \Rightarrow f$ is a projection

2. $\sum_{f \in C_k} \omega(f) = 0$

We can view a weighting $\omega$ as a linear combination of operations $f$ that have nonzero weight $\omega(f)$.

$$\omega = a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4$$
Weightings

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  Let $C \subseteq \text{Op}_D$ be a clone, $C_k$ denotes $k$-ary operations in $C$. Mapping $\omega : C_k \rightarrow \mathbb{Q}$ is a **weighting** if
  
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Weightings

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Weighted polymorphism

- Polymorphism $f$ of a classic relation $R^C$ induced by weighted relation $R$:

$$
\begin{bmatrix}
R_C & R_C & R_C \\
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23}
\end{bmatrix} \implies 
\begin{bmatrix}
R_C \\
f(x_{11}, x_{12}, x_{13}) \\
f(x_{21}, x_{22}, x_{23})
\end{bmatrix} = 
\begin{bmatrix}
f(x_{1*}) \\
f(x_{2*})
\end{bmatrix}
$$

- Weighted polymorphism $\omega = a_1 f_1 + a_2 f_2 + a_3 f_3$ of $R$ is a weighting of a clone $E \subseteq \text{Pol} \{R^C\}$:

$$
\begin{bmatrix}
R_C & R_C & R_C \\
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} 
\end{bmatrix} \implies
\begin{bmatrix}
R \\
f_1(x_{1*}) \\
f_2(x_{2*}) \\
f_3(x_{2*})
\end{bmatrix} \leq 0
$$

$R$ is improved by $\omega$. 
Weighted polymorphism

- Polymorphism $f$ of a classic relation $R^C$ induced by weighted relation $R$:
  \[
  \begin{array}{ccc}
  \overline{R^C} & \overline{R^C} & \overline{R^C} \\
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23}
  \end{array}
  \Rightarrow
  \begin{array}{c}
  \overline{R^C} \\
  f(x_{11}, x_{12}, x_{13})
  \end{array}
  =
  \begin{array}{c}
  \overline{R^C} \\
  f(x_{21}, x_{22}, x_{23})
  \end{array}
  =
  \begin{array}{c}
  f(x_{1}^*) \\
  f(x_{2}^*)
  \end{array}
  =
  f(x_1^*)
  \]

- Weighted polymorphism $\omega = a_1 f_1 + a_2 f_2 + a_3 f_3$ of $R$ is a weighting of a clone $E \subseteq \text{Pol}(\{R^C\})$:
  \[
  \begin{array}{ccc}
  \overline{R^C} & \overline{R^C} & \overline{R^C} \\
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23}
  \end{array}
  \Rightarrow
  \begin{array}{c}
  \overline{R} \\
  f_1(x_{1}^*)
  \end{array}
  +
  \begin{array}{c}
  \overline{R} \\
  f_2(x_{2}^*)
  \end{array}
  +
  \begin{array}{c}
  \overline{R} \\
  f_3(x_{2}^*)
  \end{array}
  \leq 0
  \]

$R$ is improved by $\omega$. 

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Galois correspondence
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\[ wOp_D \]  \[ wRel_D \]

\[ wPol(R) \]  \[ Pol(R) \]

\[ R \]

\[ Op_D \]
Galois correspondence

\[ \text{wOp}_D \to \text{wRel}_D \]

\[ \text{Op}_D \]

\[ \omega_1 \quad \omega_2 \]

\[ c_1 \cap c_2 \]
Galois correspondence

\[ \text{Op}_D \quad \text{wOp}_D \quad \text{wRel}_D \]

\[
\begin{array}{c}
\text{Clone} \{C_1, C_2\} \\
\text{Op}_D \\
\omega_1, \omega_2
\end{array}
\]
Galois correspondence

Weighted clones

VCSP

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Weighted clones and weighted relational clones

\( \mathcal{F} \subseteq \text{wOp}_D \); \( \text{wClone}(\mathcal{F}) \):
- contains zero weightings of every arity \( \omega(f) = 0 \)
- closed under nonegative multiplication and addition of weightings
- closed under proper superposition

\[ \begin{align*}
\text{ternary} \ & \omega = a_1 f_1 + a_2 f_2, \\
\text{binary} \ & \omega[g_1, g_2, g_3] = a_1 f_1[g_1, g_2, g_3] + a_2 f_2[g_1, g_2, g_3].
\end{align*} \]
(No negative weight on nonprojection)

\( \mathcal{R} \subseteq \text{wRel}_D \); \( \text{wRelClone}(\mathcal{R}) \):
- contains weighted equality relation
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\[ S(x_1, x_3, x_4) := \min_{x_2} (R_1(x_1, x_2, x_3) + R_2(x_2, x_4)) \]
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Weighted clones and weighted relational clones

\[ \mathcal{F} \subseteq wOp_D ; \quad \text{wClone}(\mathcal{F}) : \]

- contains zero weightings of every arity \( \omega(f) = 0 \)
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\[
\begin{align*}
\text{ternary } \omega & = a_1 f_1 + a_2 f_2, \\
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Galois correspondence

\[(\text{Cohen, Cooper, Creed, Jeavons, Živný 2012})\]
Weightings that generate all weightings.

- A non-zero weighting $\omega$ with nonzero weight on projections only, WLOG $\omega(\pi_1) > 0$:
  - $\omega_0 := \frac{1}{\omega(\pi_1)} \omega[\pi_1^2, \pi_2^2, \pi_2^2, ...] = \pi_1^2 - \pi_2^2$
  - $\delta = a_1 f_1 + a_2 f_2 + ... + a_n f_n$
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- A nonzero weighting $\omega$ with a positive weight on a projection.

- Generate binary weightings, whose weights on nonprojection operations forms a strictly diagonally dominant matrix.
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Structure of weighted clones on $D = \{0, 1\}$

Weighted clones over Clone($\{\land, \lor\}$)

- We can classify the binary parts of all weighted clones.
- WLOG we can consider only weightings of the form

$$\omega = a\pi_1 + (-1 - a)\pi_2 + b \land + (1 - b)\lor$$

$$a \in [-1, 0], b \in [0, 1]$$

- Such weighting can be visualized in an $ab$ plane:
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Thank you.