On the Minimal Arity of a Near Unanimity Operation in a Clone

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Outline

1. Notations and Definitions
2. Minimal Arity of NU
3. Criteria of Existence NU
4. Proof for the Idempotent Case
Main notations.

Let $A$ be a finite set. Everybody knows what clone is...

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- $O^m_A$ is the set of all operations on $A$ of arity at most $m$. 
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- $[M]$ is the least clone containing $M$.
- $\text{Pol}(\rho)$ is the set of all operations preserving the relation $\rho$.
- $\text{Pol}(F)$ is the set of all operations preserving every relation from $F$. 
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- $\text{Pol}(F)$ is the set of all operations preserving every relation from $F$.
- $\langle \rho \rangle_C$ is the least relation preserved by a clone $C$ and containing $\rho$. 
Near-unanimity operation

**Definition**

A near unanimity operation (NU) is an operation $f$ satisfying

$$f(x, \ldots, x, y) = f(x, \ldots, x, y, x) = \cdots = f(y, x, \ldots, x) = x.$$
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**Properties**

- Every clone with a NU operation of arity $n$ is defined by relations of arity at most $n - 1$ (K. A. Baker, A. F. Pixley).
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- H. Lakser and S. Kerkhoff successfully studied the order of clones with NU (the minimal arity of a generating set).
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- For any NU there exists finitely many clones containing it.
- Every clone with a NU operation is finitely generated.
- H. Lakser and S. Kerkhoff successfully studied the order of clones with NU (the minimal arity of a generating set).
- Sublattices of clones containing a NU can be computed by a computer (we found 1,918,040 clones containing a majority operation on 3 elements, and so on).
Two decision problems

**Question**

How can we recognize that a clone contains a NU?
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Problem 1
Given a finite set of operations $M$, decide whether $[M]$ contains a NU operation.

Problem 2
Given a finite set of relations $F$, decide whether $\text{Pol}(F)$ contains a NU operation.
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Given a finite set of operations $M$, decide whether $[M]$ contains a NU operation.

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Given a finite set of relations $F$, decide whether $\text{Pol}(F)$ contains a NU operation.

- Note that for any fixed $n$ we can easily check if a clone contains a NU of arity $n$.
- Thus, to solve Problem 1 and Problem 2 we just need an upper bound on the minimal arity of a NU.
**Definition**

\( NU(C) \) denotes the minimal arity of a NU in a clone \( C \).
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\[
NU_{\text{Oper}}(m) = \max\{NU(C) \mid C = [M], M \subseteq O_A^m, NU(C) < \infty\},
\]
\[
NU_{\text{Rel}}(m) = \max\{NU(C) \mid C = \text{Pol}(F), F \subseteq R_A^m, NU(C) < \infty\}.
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**Fact**

- Problem 1 is decidable if and only if \( \text{NU}_{\text{Oper}}(m) \) is computable.
- Problem 2 is decidable if and only if \( \text{NU}_{\text{Rel}}(m) \) is computable.
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- Problem 1 is decidable if and only if $NU_{\text{Oper}}(m)$ is computable.
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For idempotent operations $IO_A$ and conservative operations $CO_A$ we put

$NU_{\text{IdemOper}}(m) = \max\{NU(C) \mid C = [M], M \subseteq IO_A^m, NU(C) < \infty\},$

$NU_{\text{ConsOper}}(m) = \max\{NU(C) \mid C = [M], M \subseteq CO_A^m, NU(C) < \infty\}.$
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Theorem

$NU_{\text{Oper}}(m) \leq |A|^2 \cdot (|A| \cdot m)^{(3|A|)|A|}$.
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1. $NU_{Oper}(m) \leq |A|^2 \cdot (|A| \cdot m)^{|3|A|}|A|$. 
2. $NU_{IdemOper}(m) \leq m \cdot |A|^3$. 
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- L. Barto showed that $NU_{Rel}(m) \leq 4^{|A|^m}$. 
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What was known

- L.Barto proved Zadori Conjecture which implies the decidability of Problem 2.
- L.Barto showed that $\text{NU}_{\text{Rel}}(m) \leq 4^{8|A|^m}$.

Theorem

$\text{NU}_{\text{Rel}}(m) \leq ((|A| - 1)(m - 1))^{3|A|} + 1.$
Problem 2: Given a relation, decide whether it admits a NU operation

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$\text{NU}_{\text{Rel}}(m) \leq ((|A| - 1)(m - 1))^{3|A|} + 1.$

Theorem

1. $\text{NU}_{\text{Rel}}(m) \geq (m - 1)^{2|A|^2 - 2}$.
2. $\text{NU}_{\text{Rel}}(2) \geq 2^{2|A|^3}$ for $|A| \geq 4$. 
Further Investigations

Theorem

1. \( ?? \leq NU_{\text{Oper}}(m) \leq |A|^2 \cdot (|A| \cdot m)^{|A|}. \)
2. \( ?? \leq NU_{\text{IdemOper}}(m) \leq m \cdot |A|^3. \)
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**Theorem**

1. \[ (m - 1)^{2|A| - 2} \leq NU_{\text{Rel}}(m) \leq ((|A| - 1)(m - 1))^{3|A|} + 1. \]
2. \[ 2^{2|A| - 3} \leq NU_{\text{Rel}}(2) \leq (|A| - 1)^{3|A|} + 1. \]
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### Complexity of the algorithms

#### Question

Can we really use these estimates to solve Problem 1 and Problem 2?

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- To check that binary relations on 3 elements admit a NU we need to check all NU of arity $4^{3^3}$. 
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- To check that binary relations on 3 elements admit a NU we need to check all NU of arity $4^{3^3}$.
- To check that binary operations on 3 elements generate a NU we need to check all NU of arity $3^2 \cdot (3 \cdot 2)^{93}$.
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**Examples**
- To check that binary relations on 3 elements admit a NU we need to check all NU of arity $4^{3^3}$.
- To check that binary operations on 3 elements generate a NU we need to check all NU of arity $3^2 \cdot (3 \cdot 2)^{9^3}$

Can we do better?
Criteria of existence NU for idempotent case

Definition

\[ \text{Block}(D, B) = \bigcap_{n=1}^{\infty} \text{Pol}(D^n \setminus (D \setminus B)^n) \text{ for } B \subset D \subseteq A. \]

\[ \text{Lin}(D, \varphi) = \text{Pol}\{(x_1, x_2, x_3, x_4) \mid \varphi(x_1) + \varphi(x_2) = \varphi(x_3) + \varphi(x_4)\} \]

for \( D \subseteq A \), a finite field \( F \) and a surjective mapping \( \varphi : D \rightarrow F \).

- \( \text{Lin}(D, \varphi) \) is the clone of all operations that are linear on \( D \) with respect to \( \varphi \).
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for \(D \subseteq A\), a finite field \(F\) and a surjective mapping \(\varphi : D \rightarrow F\).

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Theorem

An idempotent clone \(C\) doesn’t contain a NU iff

1. \(C \subseteq \text{Block}(B, D)\) for some \(B, D\), or
2. \(C \subseteq \text{Lin}(D, \varphi)\) for some \(D\) and \(\varphi\).
Criteria of existence NU for idempotent case

**Corollary**

\[ \text{Block}(B, D) \text{ and } \text{Lin}(D, \varphi) \text{ are maximal idempotent clones that do not contain a NU.} \]
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Corollary

$\text{Block}(B, D)$ and $\text{Lin}(D, \varphi)$ are maximal idempotent clones that do not contain a NU.

How can we check whether a finite set of idempotent operations generate a NU?
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**Corollary**

\(\text{Block}(B, D)\) and \(\text{Lin}(D, \varphi)\) are maximal idempotent clones that do not contain a NU.

**How can we check whether a finite set of idempotent operations generate a NU?**  
- We check whether \(M \subseteq \text{Block}(D, B)\) for some \(B \subseteq D \subseteq A\).  
  Note that \(f \in \text{Block}(D, B)\) iff \(f\) preserves \(D\) and for some variable \(i\) we have \(f(D, D, \ldots, D, B, D, \ldots, D) \subseteq B\).
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So, it is easy!
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- We check that $M \subseteq Lin(D, ϕ)$. 
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  So, it is easy!

- We check that $M \subseteq Lin(D, \varphi)$. Easy!

  This idea can be generalized for nonidempotent case.
We need to prove that any clone generated by operations of arity at most $m$ either contains NU of arity $m \cdot |A|^3$, or doesn’t contain NU at all.
Sketch proof

Let $C$ be a clone generated by operations of arity at most $m$ and $NU(C) = n + 1$. 

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   We put $D = \langle \{a, b\} \rangle_C$ and $B_1 = \{b\}$.

2. We build a sequence $(n_1, B_1), (n_2, B_2), (n_3, B_3), \ldots$
   satisfying $\{a\}^{n_i} \not\subseteq \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C$ and
   $n_{i+1} \geq n_i - (m - 1), B_i \subset B_{i+1}$. 
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Let \( C \) be a clone generated by operations of arity at most \( m \) and \( NU(C) = n + 1 \).

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2. We build a sequence \( (n_1, B_1), (n_2, B_2), (n_3, B_3), \ldots \) satisfying \( \{a\}^{n_i} \not\in \langle D^{n_i} \setminus (D \setminus B_i)^{n_i}\rangle_C \) and \( n_{i+1} \geq n_i - (m - 1), B_i \subset B_{i+1} \).

3. We finish the sequence if \( C \) preserves \( D^{n_i} \setminus (D \setminus B_i)^{n_i} \). Since the sequence has at most \( |A| - 1 \) elements, we get \( n_i > n/k^2 - (|A| - 1) \cdot (m - 1) \).
Sketch proof

Let $C$ be a clone generated by operations of arity at most $m$ and $NU(C) = n + 1$.

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3. We finish the sequence if $C$ preserves $D^{n_i} \setminus (D \setminus B_i)^{n_i}$. Since the sequence has at most $|A| - 1$ elements, we get $n_i > n/k^2 - (|A| - 1) \cdot (m - 1)$. If $n_i \geq m$ then $C$ preserves $D^p \setminus (D \setminus B_i)^p$ for any $p$. Contradiction!!!
**Step 1**

*C* contains a NU of arity \( n + 1 \) but not \( n \).

**We know from AAA87**

1. A clone *C* doesn’t contain a NU of arity \( n \) iff there exists a compatible key (critical) relation of arity \( n \).

2. For any key relation \( \rho \) preserved by a NU we can find \((a_1, a_2, \ldots, a_n) \notin \rho\) such that
\[
\left( \{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_n, b_n\} \right) \setminus \{(a_1, a_2, \ldots, a_n)\} \subseteq \rho.
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\]

Thus, we have $\rho$ preserved by $C$, $(a_1, a_2, \ldots, a_n) \notin \rho$ and

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\left(\{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_n, b_n\}\right) \setminus \{(a_1, a_2, \ldots, a_n)\} \subseteq \rho.
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   ([a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]) \setminus \{(a_1, a_2, \ldots, a_n)\} \subseteq \rho.
   \]

Thus, we have \(\rho\) preserved by \(C\), \((a_1, a_2, \ldots, a_n) \notin \rho\) and
   \[
   ([a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]) \setminus \{(a_1, a_2, \ldots, a_n)\} \subseteq \rho.
   \]

\((a_1, \ldots, a_n) \notin \langle([a_1, b_1] \times \cdots \times [a_n, b_n]) \setminus \{(a_1, \ldots, a_n)\}\rangle_C.\]
Step 1

$C$ contains a NU of arity $n + 1$ but not $n$.

**We know from AAA87**

1. A clone $C$ doesn’t contain a NU of arity $n$ iff there exists a compatible key (critical) relation of arity $n$.

2. For any key relation $\rho$ preserved by a NU we can find $(a_1, a_2, \ldots, a_n) \notin \rho$ such that
   \[
   (\{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_n, b_n\}) \setminus \{(a_1, a_2, \ldots, a_n)\} \subseteq \rho.
   \]

Thus, we have $\rho$ preserved by $C$, $(a_1, a_2, \ldots, a_n) \notin \rho$ and
\[
(\{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_n, b_n\}) \setminus \{(a_1, a_2, \ldots, a_n)\} \subseteq \rho.
\]

\[
(a_1, \ldots, a_n) \notin \langle(\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}) \setminus \{(a_1, \ldots, a_n)\}\rangle_C.
\]

We consider idempotent case!
Step 1

C contains a NU of arity $n + 1$ but not $n$.

We know from AAA87

1. A clone C doesn’t contain a NU of arity $n$ iff there exists a compatible key (critical) relation of arity $n$.

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   $$(\{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_n, b_n\}) \setminus \{(a_1, a_2, \ldots, a_n)\} \subseteq \rho.$$ 

Thus, we have $\rho$ preserved by $C$, $(a_1, a_2, \ldots, a_n) \notin \rho$ and

$$(\{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_n, b_n\}) \setminus \{(a_1, a_2, \ldots, a_n)\} \subseteq \rho.$$ 

$$(a_1, \ldots, a_n) \notin \langle \{(a_1, b_1) \times \cdots \times \{a_n, b_n\}) \setminus \{(a_1, \ldots, a_n)\} \rangle_C.$$ 

We consider idempotent case! We choose the most popular pair $(a_i, b_i)$ to get $\{a\}^{n_1} \notin \langle \{a, b\}^{n_1} \setminus \{a\}^{n_1} \rangle_C$ for $n_1 \geq n/|A|^2$. 
Step 2

We have $n_1$ and $\{a\}^{n_1} \notin \langle \{a, b\}^{n_1} \setminus \{a\}^{n_1} \rangle_c$. 
Step 2

We have $n_1$ and $\{a\}^{n_1} \notin \langle \{a, b\}^{n_1} \setminus \{a\}^{n_1} \rangle_C$.

For $D = \langle \{a, b\} \rangle_C$ and $B_1 = \{b\}$ we have

$\{a\}^{n_1} \notin \langle D^{n_1} \setminus (D \setminus B_1)^{n_1} \rangle_C \subseteq \langle \{a, b\}^{n_1} \setminus \{b\}^{n_1} \rangle_C$.

We have the first element of the sequence: $(n_1, B_1)$. 
Step 2

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Suppose we have $(n_i, B_i)$. If $C$ preserves $D^{n_i} \setminus (D \setminus B_i)^{n_i}$ we are done.
Step 2

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Suppose we have $(n_i, B_i)$. If $C$ preserves $D^{n_i} \setminus (D \setminus B_i)^{n_i}$ we are done. If not, then an operation $f \in C$ doesn’t preserve $D^{n_i} \setminus (D \setminus B_i)^{n_i}$.

\[
\begin{pmatrix}
\vdots \\
\vdots \\
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\begin{pmatrix}
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\end{pmatrix}
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\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
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\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}

f

\begin{pmatrix}
\vdots \\
\vdots \\
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\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
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\end{pmatrix}
\begin{pmatrix}
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\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}

\notin D^{n_i} \setminus (D \setminus B_i)^{n_i}.
Step 2

We have \( n_1 \) and \( \{a\}^{n_1} \not\in \langle \{a, b\}^{n_1} \setminus \{a\}^{n_1} \rangle_C \).

For \( D = \langle \{a, b\} \rangle_C \) and \( B_1 = \{b\} \) we have

\[
\{a\}^{n_1} \not\in \langle D^{n_1} \setminus (D \setminus B_1)^{n_1} \rangle_C \subseteq \langle \{a, b\}^{n_1} \setminus \{b\}^{n_1} \rangle_C.
\]

We have the first element of the sequence: \((n_1, B_1)\).

Suppose we have \((n_i, B_i)\). If \( C \) preserves \( D^{n_i} \setminus (D \setminus B_i)^{n_i} \) we are done. If not, then an operation \( f \in C \) doesn’t preserve \( D^{n_i} \setminus (D \setminus B_i)^{n_i} \).

\[
f\begin{pmatrix}
\begin{pmatrix}
\cdot
& \cdot
& \cdot
& \cdots

\cdot
& \cdot
& \cdot
& \cdots

\cdot
& \cdot
& \cdot
& \cdots

\cdots
& \cdots
& \cdots
& \cdots
\end{pmatrix}
&
\begin{pmatrix}
\cdot
& \cdot
& \cdot
& \cdots

\cdot
& \cdot
& \cdot
& \cdots

\cdot
& \cdot
& \cdot
& \cdots

\cdots
& \cdots
& \cdots
& \cdots
\end{pmatrix}
&
\begin{pmatrix}
\cdot
& \cdot
& \cdot
& \cdots

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& \cdot
& \cdot
& \cdots

\cdot
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& \cdots

\cdots
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& \cdots
& \cdots
\end{pmatrix}
&
\begin{pmatrix}
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& \cdot
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& \cdot
& \cdots

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& \cdot
& \cdot
& \cdots

\cdots
& \cdots
& \cdots
& \cdots
\end{pmatrix}
\end{pmatrix}
\not\in D^{n_i} \setminus (D \setminus B_i)^{n_i}.
\]
We have $n_1$ and $\{a\}^{n_1} \not\in \langle \{a, b\}^{n_1} \setminus \{a\}^{n_1} \rangle_C$.

For $D = \langle \{a, b\} \rangle_C$ and $B_1 = \{b\}$ we have

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Suppose we have $(n_i, B_i)$. If $C$ preserves $D^{n_i} \setminus (D \setminus B_i)^{n_i}$ we are done. If not, then an operation $f \in C$ doesn’t preserve $D^{n_i} \setminus (D \setminus B_i)^{n_i}$.

$$f \begin{pmatrix} (\ldots) \\ b \\ (\ldots) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \not\in D^{n_i} \setminus (D \setminus B_i)^{n_i}.$$
Step 2

We have $n_1$ and $\{a\}^{n_1} \notin \langle \{a, b\}^{n_1} \setminus \{a\}^{n_1} \rangle_C$.

For $D = \langle \{a, b\} \rangle_C$ and $B_1 = \{b\}$ we have

$\{a\}^{n_1} \notin \langle D^{n_1} \setminus (D \setminus B_1)^{n_1} \rangle_C \subseteq \langle \{a, b\}^{n_1} \setminus \{b\}^{n_1} \rangle_C$.

We have the first element of the sequence: $(n_1, B_1)$.

Suppose we have $(n_i, B_i)$. If $C$ preserves $D^{n_i} \setminus (D \setminus B_i)^{n_i}$ we are done. If not, then an operation $f \in C$ doesn’t preserve $D^{n_i} \setminus (D \setminus B_i)^{n_i}$.

$$
\begin{pmatrix}
  \vdots \\
  b \\
  \vdots \\
  a \\
  a \\
  a \\
  a \\
\end{pmatrix}
\begin{pmatrix}
  \vdots \\
  \vdots \\
  \vdots \\
  a \\
  a \\
  a \\
  a \\
\end{pmatrix}
\begin{pmatrix}
  \vdots \\
  b \\
  \vdots \\
  a \\
  a \\
  a \\
  a \\
\end{pmatrix}
= \begin{pmatrix}
  c_1 \\
  c_2 \\
  c_3 \\
  c_4 \\
  a \\
  a \\
  a \\
\end{pmatrix}

\notin D^{n_i} \setminus (D \setminus B_i)^{n_i}.$$

$$
\begin{pmatrix}
  \vdots \\
  b \\
  \vdots \\
  a \\
  a \\
  a \\
  a \\
\end{pmatrix}
\begin{pmatrix}
  \vdots \\
  \vdots \\
  \vdots \\
  a \\
  a \\
  a \\
  a \\
\end{pmatrix}
\begin{pmatrix}
  \vdots \\
  b \\
  \vdots \\
  a \\
  a \\
  a \\
  a \\
\end{pmatrix}
= \begin{pmatrix}
  c_1 \\
  c_2 \\
  c_3 \\
  c_4 \\
  a \\
  a \\
  a \\
\end{pmatrix}

\notin D^{n_i} \setminus (D \setminus B_i)^{n_i}.$$
Step 2

We get $k \leq m$ such that
\[(c_1, \ldots, c_k, a, \ldots, a) \in \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C\]
and
\[(c_1, \ldots, c_k, a, \ldots, a) \notin D^{n_i} \setminus (D \setminus B_i)^{n_i}.\]
We get $k \leq m$ such that
\[(c_1, \ldots, c_k, a, \ldots, a) \in \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C \]
and
\[(c_1, \ldots, c_k, a, \ldots, a) \notin D^{n_i} \setminus (D \setminus B_i)^{n_i}.\]

WLOG, let $k$ be the minimal number with this property. Then
\[(c_1, \ldots, c_{k-1}, a, a, \ldots, a) \notin \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C \]
Step 2

We get $k \leq m$ such that

$$(c_1, \ldots, c_k, a, \ldots, a) \in \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C$$

and

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WLOG, let $k$ be the minimal number with this property. Then

$$(c_1, \ldots, c_{k-1}, a, a, \ldots, a) \not\in \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C$$

Since we consider idempotent case, we have

$$(c_1, \ldots, c_{k-1}, c_k, D, \ldots, D) \subseteq \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C$$
Step 2

We get $k \leq m$ such that

$$(c_1, \ldots, c_k, a, \ldots, a) \in \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C$$

and

$$(c_1, \ldots, c_k, a, \ldots, a) \notin D^{n_i} \setminus (D \setminus B_i)^{n_i}.$$  

WLOG, let $k$ be the minimal number with this property. Then

$$(c_1, \ldots, c_{k-1}, a, a, \ldots, a) \notin \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C$$

Since we consider idempotent case, we have

$$(c_1, \ldots, c_{k-1}, c_k, D, \ldots, D) \subseteq \langle D^{n_i} \setminus (D \setminus B_i)^{n_i} \rangle_C$$

Put $n_{i+1} = n_i - (k - 1)$, $B_{i+1} = B_i \cup \{c_k\}$. Then

$$\{a\}^{n_{i+1}} \notin \langle D^{n_{i+1}} \setminus (D \setminus B_{i+1})^{n_{i+1}} \rangle_C.$$  

Let $(n_{i+1}, B_{i+1})$ be the next element of the sequence.
Step 3

We finish the sequence with $D^{n_i} \setminus (D \setminus B_i)^{n_i}$ preserved by $C$. If $n_i \geq m$ then $C$ preserves $D^p \setminus (D \setminus B_i)^p$ for any $p$. But a NU cannot preserve $D^p \setminus (D \setminus B_i)^p$ for any $p$. Contradiction!!!
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Thus, $m > n_i > n/|A|^2 - (|A| - 1) \cdot (m - 1)$
Therefore, $n < |A|^3 \cdot m$.

We complete the proof
Further Investigations

**Question**
Can we recognize that a clone contains a NU?
Further Investigations

Question
Can we recognize that a clone contains a NU?

- For clones defined by finite set of operations.
Further Investigations

**Question**

Can we recognize that a clone contains a NU?

- For clones defined by finite set of operations. *Decidable!*
Further Investigations

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Can we recognize that a clone contains a NU?

- For clones defined by finite set of operations. **Decidable!**
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- For clones defined by finite set of operations. **Decidable!**
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- For clones of the form \([A] \cap [B]\) where \(A\) and \(B\) are finite sets of operations.
Further Investigations

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- For clones defined by finite set of operations. Decidable!
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What do you know about $[A] \cap [B]$?
Further Investigations

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What do you know about \([A] \cap [B]\)?

- Can you check whether it is finitely generated?
Further Investigations

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- For clones defined by finite set of operations. \textbf{Decidable}!
- For clones defined by finite set of relations. \textbf{Decidable}!
- For clones of the form \([A] \cap [B]\) where \(A\) and \(B\) are finite sets of operations.

What do you know about \([A] \cap [B]\)?

- Can you check whether it is finitely generated? \textbf{I can’t!}
Further Investigations

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Can we recognize that a clone contains a NU?

- For clones defined by finite set of operations. Decidable!
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- For clones of the form $[A] \cap [B]$ where $A$ and $B$ are finite sets of operations.

What do you know about $[A] \cap [B]$?

1. Can you check whether it is finitely generated? I can’t!
2. Can you check whether it is finitely related?
Further Investigations

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What do you know about $[A] \cap [B]$?

1. Can you check whether it is finitely generated? *I can’t!*
2. Can you check whether it is finitely related? *I can’t!*
3. Can you check that $[A] \cap [B] = [C]$?
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Can we recognize that a clone contains a NU?

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Can we recognize that a clone contains a NU?

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What do you know about \([A] \cap [B]\)?

1. Can you check whether it is finitely generated? **I can’t!**
2. Can you check whether it is finitely related? **I can’t!**
3. Can you check that \([A] \cap [B] = [C]\)? **I can’t!**
4. Can you check that \([A] \cap [B]\) contains NU?
Further Investigations

Question

Can we recognize that a clone contains a NU?

- For clones defined by finite set of operations. Decidable!
- For clones defined by finite set of relations. Decidable!
- For clones of the form \([A] \cap [B]\) where \(A\) and \(B\) are finite sets of operations.

What do you know about \([A] \cap [B]\)?

1. Can you check whether it is finitely generated? I can’t!
2. Can you check whether it is finitely related? I can’t!
3. Can you check that \([A] \cap [B] = [C]\)? I can’t!
4. Can you check that \([A] \cap [B]\) contains NU? I can!
Further Investigations

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- For clones defined by finite set of operations. **Decidable!**
- For clones defined by finite set of relations. **Decidable!**
- For clones of the form \([A] \cap [B]\) where \(A\) and \(B\) are finite sets of operations. **Decidable!**
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- For clones defined by finite set of relations. **Decidable!**
- For clones of the form $[A] \cap [B]$ where $A$ and $B$ are finite sets of operations. **Decidable!**
- For clones of the form $[A_1] \cap [A_2] \cap \cdots \cap [A_n]$ where $A_1, A_2, \ldots, A_n$ are finite sets of operations. **Decidable!**
Further Investigations

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Can we recognize that a clone contains a NU?

- For clones defined by finite set of operations. **Decidable!**
- For clones defined by finite set of relations. **Decidable!**
- For clones of the form \([A] \cap [B]\) where \(A\) and \(B\) are finite sets of operations. **Decidable!**
- For clones of the form \([A_1] \cap [A_2] \cap \cdots \cap [A_n]\) where \(A_1, A_2, \ldots, A_n\) are finite sets of operations. **Decidable!**
- For clones of the form \([\text{Pol}(\rho_1) \cup \text{Pol}(\rho_2)]\).
Further Investigations

Question

Can we recognize that a clone contains a NU?

- For clones defined by finite set of operations. Decidable!
- For clones defined by finite set of relations. Decidable!
- For clones of the form $[A] \cap [B]$ where $A$ and $B$ are finite sets of operations. Decidable!
- For clones of the form $[A_1] \cap [A_2] \cap \cdots \cap [A_n]$ where $A_1, A_2, \ldots, A_n$ are finite sets of operations. Decidable!
- For clones of the form $[\text{Pol}(\rho_1) \cup \text{Pol}(\rho_2)]$. I don’t know!

Open problem: is this problem decidable?

Given relations $\rho_1$ and $\rho_2$, decide whether $[\text{Pol}(\rho_1) \cup \text{Pol}(\rho_2)]$ contains a NU.
Thank you for your attention