ON SINGULARITIES OF HAMILTONIAN MAPPINGS

TAKUO FUKUDA
Department of Mathematics
College of Humanities and Sciences, Nihon University
3-25-40 Sakurajousui, Setagaya-ku, Tokyo, Japan

STANISŁAW JANECZKO
Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, Warszawa, Poland, and
Faculty of Mathematics and Information Science, Warsaw University of Technology
Plac Politechniki 1, 00-661 Warszawa, Poland
E-mail: janeczko@impan.gov.pl

Abstract. The notion of an implicit Hamiltonian system—an isotropic mapping $H : M \to (TM, \dot{\omega})$ into the tangent bundle endowed with the symplectic structure defined by canonical morphism between tangent and cotangent bundles of $M$—is studied. The corank one singularities of such systems are classified. Their transversality conditions in the 1-jet space of isotropic mappings are described and the corresponding symplectically invariant algebras of Hamiltonian generating functions are calculated.

1. Introduction. Let $(M, \omega)$ be a symplectic manifold. A Hamiltonian system is an isotropic section $F : M \to TM$ of the tangent bundle $TM$ endowed with the symplectic structure defined by the canonical morphism $\beta$ between tangent and cotangent bundles of $M$ appearing in the commuting diagram (cf. [16])

$$
\begin{array}{ccc}
M & \xrightarrow{F} & (TM, \dot{\omega}) \\
\downarrow \pi & & \downarrow \Pi \\
(M, \omega) & \xrightarrow{\beta} & (T^*M, d\theta)
\end{array}
$$

We have $F^*\dot{\omega} = 0$ and $\bar{F} = \pi \circ F$, $\dot{\omega} = \beta^{-1}(d\theta)$.

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If we consider $F$ to be a smooth isotropic regular mapping of another manifold $N$ $(\dim N = \dim M)$ into $TM$ then $F$ is a parametrization of a Hamiltonian system in $(TM, \dot{\omega})$, which in general if $\vec{F}$ has singularities (cf. [12]) is an implicit Hamiltonian system (see e.g. [1, 2, 5, 6, 10, 11, 15]). For each isotropic mapping $F$ there exists at least locally a generating Hamiltonian function $h : N \to \mathbb{R}$ such that $(\beta \circ F)^*\theta = -dh$. In this paper we study the symplectically invariant algebra of Hamiltonian generating functions determined by the mapping $\vec{F}$ or more precisely by its singularity type. In the corank one singularity case this algebra is defined by the ideal generated by the determinant of the Jacobi matrix of $\vec{F}$. The algebra of generating Hamiltonian functions in the general corank case singularity of $\vec{F}$ is calculated and conditions on an isotropic map $F$ ensuring that the one-jet extension $j^1F$ is transversal to the corank one stratum in the isotropic 1-jet space of mappings are derived. These conditions are obtained if, first, $\vec{F}$ has corank one singularity and then corank $k$ singularity with $k \geq 2$.

2. Isotropic mappings. Let $(\mathbb{R}^{2n}, \omega)$ be a Euclidean symplectic space, $\omega = \sum_{i=1}^{n} dy_i \wedge dx_i$ in canonical Darboux coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Let $\theta$ be the Liouville 1-form on the cotangent bundle $T^*\mathbb{R}^{2n}$. Then $d\theta$ is the standard symplectic structure on $T^*\mathbb{R}^{2n}$. Let $\beta : T\mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ be the canonical bundle map defined by $\omega$,

$$\beta : T\mathbb{R}^{2n} \ni v \mapsto \omega(v, \cdot) \in T^*\mathbb{R}^{2n}.$$ 

Then we can define the canonical symplectic structure $\dot{\omega}$ on $T\mathbb{R}^{2n}$,

$$\dot{\omega} = \beta^*d\theta = d(\beta^*\theta) = \sum_{i=1}^{n} (d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i),$$

where $(x, y, \dot{x}, \dot{y})$ are local coordinates on $T\mathbb{R}^{2n}$ and $\beta^*\theta = \sum_{i=1}^{n} (\dot{y}_i dx_i - \dot{x}_i dy_i)$.

Throughout the paper if not otherwise stated all objects are germs at 0 of smooth functions, mappings, forms etc. or their representatives on an open neighbourhood of 0 in $\mathbb{R}^{2n}$.

**Definition 2.1.** Let $F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ be a smooth map-germ. We say that $F$ is isotropic if $F^*\dot{\omega} = 0$.

**Proposition 2.2.** A smooth map-germ $F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ is isotropic if and only if there exists a smooth function-germ $h : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ such that

$$(\beta \circ F)^*\theta = -dh. \quad (2.1)$$

For each smooth isotropic map-germ $F$ such a function-germ $h$ is unique up to an additive constant.

**Proof.** We assume that $F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ is isotropic, then the differential of the 1-form $(\beta \circ F)^*\theta$ defined on some contractible open neighbourhood $U \subset \mathbb{R}^{2n}$ of 0 vanishes,

$$d(\beta \circ F)^*\theta = F^*\beta^*d\theta = F^*\dot{\omega} = 0.$$ 

Thus $(\beta \circ F)^*\theta$ is a closed 1-form on $U$. By the Poincaré Lemma, there exists a smooth function $h : \mathbb{R}^{2n} \ni U \to \mathbb{R}$ such that $(\beta \circ F)^*\theta = -dh$. For each smooth isotropic map-
germ \( F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n} \) there exists a unique (up to an additive constant) smooth function-germ \( h : (\mathbb{R}^{2n}, 0) \to \mathbb{R} \) such that (2.1) is fulfilled.

Let \((u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_n)\) denote coordinates of the source space \( U \subset \mathbb{R}^{2n} \). In local coordinates we define \( F = (f, g, \dot{f}, \dot{g}) : \mathbb{R}^{2n} \supset U \to T\mathbb{R}^{2n} \), and \( \tilde{F} = \pi \circ F = (f, g) : \mathbb{R}^{2n} \supset U \to \mathbb{R}^{2n} \), where \( \pi \) denotes the canonical projection \( \pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n} \). By

\[
J(\tilde{F}) = \left( \begin{array}{cc}
\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\
\frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}
\end{array} \right)
\]

we denote the Jacobi matrix of \( \tilde{F} \), i.e. the matrix of the tangent map \( d\tilde{F} \), and by \( I_n \) the unit matrix of dimension \( n \).

**Lemma 2.3.** A smooth map-germ \( F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n} \) is isotropic if and only if there exists a smooth function-germ \( h : (\mathbb{R}^{2n}, 0) \to \mathbb{R} \) such that

\[
\left( \begin{array}{c}
\frac{\partial h}{\partial u} \\
\frac{\partial h}{\partial v}
\end{array} \right) = \left( \begin{array}{cc}
\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\
\frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}
\end{array} \right) \left( \begin{array}{cc}
O & -I_n \\
I_n & O
\end{array} \right) \left( \begin{array}{c}
\dot{f} \\
\dot{g}
\end{array} \right),
\]

(2.2)

**Proof.** Equivalence of (2.1) and (2.2) can be verified by comparing \((\beta \circ F)^* \theta(\frac{\partial}{\partial u_i})\) and \((\beta \circ F)^* \theta(\frac{\partial}{\partial v_i})\) with \(dh(\frac{\partial}{\partial u_i}) = \frac{\partial h}{\partial u_i}\) and \(dh(\frac{\partial}{\partial v_i}) = \frac{\partial h}{\partial v_i}\) respectively. For we get

\[
(\beta \circ F)^* \theta(\frac{\partial}{\partial u_i}) = F^* \beta^* \theta(\frac{\partial}{\partial u_i}) = F^* \left( \sum_{j=1}^{n} \dot{y}_j dx_j - \dot{x}_j dy_j \right) \left( \frac{\partial}{\partial u_i} \right)
\]

\[
= \left( \sum_{j=1}^{n} \dot{y}_j dx_j - \dot{f}_j dy_j \right) \left( df \left( \frac{\partial}{\partial u_i} \right) \right) = \sum_{j=1}^{n} \left( \frac{\partial f_j}{\partial u_i} \dot{y}_j - \frac{\partial g_j}{\partial u_i} \dot{f}_j \right)
\]

and in the same way

\[
(\beta \circ F)^* \theta(\frac{\partial}{\partial v_i}) = \sum_{j=1}^{n} \left( \frac{\partial f_j}{\partial v_i} \dot{y}_j - \frac{\partial g_j}{\partial v_i} \dot{f}_j \right).
\]

In general \( F \) can be regarded as a vector field along \( \tilde{F} \), i.e. a section of the induced fiber bundle \( F^* T\mathbb{R}^{2n} \). By \( \mathcal{E}_U \) (\( \mathcal{E}_{\mathbb{R}^{2n}} \)-respectively) we denote the \( \mathbb{R} \)-algebra of smooth function germs at 0 on \( U \subset \mathbb{R}^{2n} \) (and on "the target space" \( \mathbb{R}^{2n} \) respectively). From Proposition 2.2, for each isotropic map-germ \( F \) along \( \tilde{F} \) there exists a unique \( h \) belonging to the maximal ideal \( m_U \) of \( \mathcal{E}_U \), which we call a generating function-germ for \( F \).

Let \( F : (U, 0) \to T\mathbb{R}^{2n} \) and \( G : (U, 0) \to T\mathbb{R}^{2n} \) be two isotropic map-germs along \( \tilde{F} : (U, 0) \to \mathbb{R}^{2n} \) and \( \tilde{G} : (U, 0) \to \mathbb{R}^{2n} \) respectively. Now we introduce the natural equivalence groups acting on isotropic mappings through a natural lifting of diffeomorphic or symplectomorphic equivalences of \( \tilde{F} \) and \( \tilde{G} \) (cf. [8, 9]).

**Definition 2.4.** 1. Let \( F : (U, 0) \to T\mathbb{R}^{2n} \) and \( G : (U, 0) \to T\mathbb{R}^{2n} \) be two isotropic map-germs. We say that \( F \) and \( G \) are Lagrangian equivalent if there exist a diffeomorphism-germ \( \varphi : (U, 0) \to (U, 0) \), and a symplectomorphism-germ \( \Psi : (T\mathbb{R}^{2n}, 0) \to (T\mathbb{R}^{2n}, 0) \), \( \Psi^* \omega = \tilde{\omega} \), preserving the fibering \( \pi \) such that \( G = \Psi \circ F \circ \varphi \).
2. Let \( F : (U, 0) \to T\mathbb{R}^{2n} \) and \( G : (U, 0) \to T\mathbb{R}^{2n} \) be two isotropic map-germs along \( \bar{F} : (U, 0) \to \mathbb{R}^{2n} \) and \( \bar{G} : (U, 0) \to \mathbb{R}^{2n} \) respectively. We say that \( F \) and \( G \) are L-symplectic equivalent if there exist a diffeomorphism-germ \( \varphi : (U, 0) \to (U, 0) \), and a symplectomorphism-germ \( \Psi : (T\mathbb{R}^{2n}, 0) \to (T\mathbb{R}^{2n}, 0) \), \( \Psi^*\dot{\omega} = \dot{\omega} \), preserving the fibering \( \pi \) and a symplectomorphism-germ \( \Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0) \), \( \Phi^*\omega = \omega \), \( \pi \circ \Psi = \Phi \circ \pi \), such that \( G = \Psi \circ F \circ \varphi \) and \( \bar{G} = \Phi \circ \bar{F} \circ \varphi \). In this case we call \( \bar{F} \) and \( \bar{G} \) symplectomorphic or symplectically equivalent.

To \( \bar{F} \) we associate the symplectically invariant algebra \( \mathcal{R}_{\bar{F}} \) of all generating function-germs (cf. [8]),

\[
\mathcal{R}_{\bar{F}} = \{ h \in \mathcal{E}_U : h \text{ generates an isotropic map-germ along } \bar{F} \} = \{ h \in \mathcal{E}_U : dh \in \mathcal{E}_U d(F^*\mathcal{E}_{\mathbb{R}^{2n}}) \}.
\]

It is easy to check that if \( \bar{F} \) has a maximal rank then \( \mathcal{R}_{\bar{F}} = \mathcal{E}_U \). The aim of this section is to study the case when \( \bar{F} \) has no maximal rank and establish the structure of \( \mathcal{R}_{\bar{F}} \).

Now we assume that \( \bar{F} \) is a corank one map-germ at \( 0 \in U \). Let \( e \in T_0 U \) span the kernel of the Jacobi matrix \( J(\bar{F}) \) at zero. By \( \Delta_{\bar{F}} \) we denote the determinant of \( J(\bar{F}) \) and by \( \partial_e \) the derivation in \( e \)-direction.

**Theorem 2.5.** Let \( F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n} \) be a smooth map-germ such that \( \bar{F} \) has corank one singularity at \( 0 \).

1. If \( F \) is isotropic then there exists a unique generating function-germ \( h : (\mathbb{R}^{2n}, 0) \to \mathbb{R} \), \( h(0) = 0 \) such that \( \partial_e h \in \langle \Delta_{\bar{F}} \rangle \), where \( \langle \Delta_{\bar{F}} \rangle \) is the ideal generated by \( \Delta_{\bar{F}} \) in \( \mathcal{E}_{\mathbb{R}^{2n}} \).

2. Conversely, for every smooth function-germ \( h : (\mathbb{R}^{2n}, 0) \to \mathbb{R} \) such that \( \partial_e h \in \langle \Delta_{\bar{F}} \rangle \) there is a unique isotropic map-germ \( F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n} \) such that \( \bar{F} = \pi \circ F \) and \( (\beta \circ F)^*\theta = -dh \).

**Proof.** Since we have assumed that the corank of \( \bar{F} = (f, g) : (U, 0) \to \mathbb{R}^{2n} \) is one at the origin, we can choose coordinates in \( U \) such that

\[
\begin{align*}
  f_i(u, v) &= u_i, & i &= 1, \ldots, n, \\
  g_i(u, v) &= v_i, & i &= 1, \ldots, n - 1, \\
  \frac{\partial g_n}{\partial v_n}(0, 0) &= 0,
\end{align*}
\]

and \( e = \frac{\partial}{\partial v_n} \). Then

\[
J(\bar{F}) = \begin{pmatrix}
  I_n & O & 0 \\
  O & I_{n-1} & 0 \\
  \frac{\partial g_n}{\partial u} & \frac{\partial g_n}{\partial v} & \frac{\partial g_n}{\partial v_n}
\end{pmatrix}
\]

where \( \bar{v} = (v_1, \ldots, v_{n-1}) \).
Since \( \dot{f}, \dot{g} \) in the equation (2.2) are smooth, we can write equivalently

\[
\begin{pmatrix}
\dot{f} \\
\dot{g}
\end{pmatrix} = \begin{pmatrix}
O & I_n \\
-I_n & O
\end{pmatrix}^t \begin{pmatrix}
\frac{\partial f}{\partial u} & \frac{\partial g}{\partial u} \\
\frac{\partial g}{\partial v} & \frac{\partial g}{\partial v}
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{\partial h}{\partial u} \\
\frac{\partial h}{\partial v}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
O & I_n \\
-I_n & O
\end{pmatrix} \begin{pmatrix}
I_n & O & -\frac{\partial g_a}{\partial u} / \Delta_F \\
O & I_{n-1} & -\frac{\partial g_a}{\partial v} / \Delta_F \\
0 & 0 & 1/\Delta_F
\end{pmatrix} \begin{pmatrix}
\frac{\partial h}{\partial u} \\
\frac{\partial g_a}{\partial u} / \Delta_F \\
\frac{\partial g_a}{\partial v} / \Delta_F \\
\frac{\partial h}{\partial v} / \Delta_F
\end{pmatrix}.
\]

(2.4)

Thus in order that the right hand side of (2.4) should be smooth we get

\[
\frac{\partial h}{\partial v_n} \in \langle \Delta_F \rangle
\]

(2.5)

so we proved item 1. If we have \( h \) which fulfills the condition (2.5) then by the formula

\[
\begin{pmatrix}
\dot{f} \\
\dot{g}
\end{pmatrix} = \begin{pmatrix}
O & I_n \\
-I_n & O
\end{pmatrix}^t \begin{pmatrix}
\frac{\partial f}{\partial u} & \frac{\partial g}{\partial u} \\
\frac{\partial g}{\partial v} & \frac{\partial g}{\partial v}
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{\partial h}{\partial u} \\
\frac{\partial h}{\partial v}
\end{pmatrix}
\]

we construct \( F \) in a unique way. \( \blacksquare \)

**Corollary 2.6.** The algebra \( \mathcal{R}_F \) of all generating function-germs (which is also an \( \mathcal{E}_{\mathbb{R}^{2n}} \)-module) for a smooth map \( \bar{F} \) of corank one, at the origin, is given by

\[
\mathcal{R}_F = \{ h \in \mathcal{E}_U : \partial_v h \in \langle \Delta_F \rangle \}.
\]

If \( \bar{F} : (U, 0) \rightarrow (\mathbb{R}^{2n}, 0) \) is a corank 1 \( C^\infty \) stable map-germ, then by B. Morin's classification theorem [14] \( \bar{F} \) is diffeomorphic to one of the following so-called \( A_k \) type singularities, \( 0 < k < 2n \):

\[
(u_1, \ldots, u_{2n}) \mapsto (u_1, \ldots, u_{2n-1}, u_{2n}^{k+1} + \sum_{i=1}^{k-1} u_i u_{2n}^{k-i})
\]

(2.6)

We call a \( C^\infty \) stable map-germ diffeomorphic to the normal form of \( A_k \) type singularity also an \( A_k \) type singularity. In this note, we classify corank 1 \( C^\infty \) stable map-germs \( \bar{F} : (U, 0) \rightarrow (\mathbb{R}^{2n}, 0) \) under the symplectomorphic equivalence.

**Theorem 2.7.** Let \( \bar{F} : (U, 0) \rightarrow (\mathbb{R}^{2n}, 0) \) be an \( A_k \) type singularity. Then \( \bar{F} \) is symplectomorphic to the following map-germ:

\[
u = (u_1, \ldots, u_{2n}) \mapsto (u_1, \ldots, u_{2n-1}, u_{2n}^{k+1} + \sum_{i=1}^{k-1} a_i(u) u_{2n}^{k-i}),
\]

(2.7)

where \( a_1(u), \ldots, a_{k-1}(u) \) are smooth function-germs such that \( da_1, \ldots, da_{k-1} \) and \( du_{2n} \) are linearly independent at the origin.

**Proof.** Let \( \bar{F} : (U, 0) \rightarrow (\mathbb{R}^{2n}, 0) \) be an \( A_k \) type singularity. Let \( (w_1, \ldots, w_{2n}) \) denote the standard coordinates in \( U \). Then there exist diffeomorphism-germs \( \phi = (\phi_1, \ldots, \phi_{2n}) : \)
we have
\[ \psi_i \circ \tilde{F} \circ \phi(w_1, \ldots, w_{2n}) = w_i, \quad i = 1, 2, \ldots, 2n - 1, \]
\[ \psi_{2n} \circ \tilde{F} \circ \phi(w_1, \ldots, w_{2n}) = w_{2n}^{k+1} + \sum_{i=1}^{k-1} w_i w_{2n}^{k-i}. \quad (2.8) \]

Since \( d\psi_{2n} \) does not vanish at the origin, there exists a symplectic coordinate system
\( (\varphi_1, \ldots, \varphi_{2n} = \psi_{2n}) \) with \( \varphi_2n = \psi_2n \). Set
\[ u_i = \varphi_i \circ \tilde{F} \circ \phi(w_1, \ldots, w_{2n}), \quad i = 1, 2, \ldots, 2n - 1, \]
\[ u_{2n} = w_{2n}. \quad (2.9) \]

Note that the map-germ \( \Phi = ((\varphi_1, \ldots, \varphi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0) \) is a symplectomorphism.

Now we show that \( (u_1, \ldots, u_{2n}) \) is a new coordinate system in \( (U, 0) \). In fact for functions \( \alpha_1, \ldots, \alpha_k \) and variables \( w_1, \ldots, w_m \), let us denote the Jacobian matrix at the origin of \( \alpha_1, \ldots, \alpha_k \) with respect to \( w_1, \ldots, w_m \) by
\[ J \left( \frac{\alpha_1, \ldots, \alpha_k}{w_1, \ldots, w_m} \right)(0). \]

Then, since both of \( \psi = (\psi_1, \ldots, \psi_{2n}) \) and \( (\varphi_1, \ldots, \varphi_{2n}) \) are local coordinate systems in the target space, we have
\[ \text{rank } J \left( \frac{\varphi_1 \circ \tilde{F} \circ \phi, \ldots, \varphi_{2n} \circ \tilde{F} \circ \phi}{w_1, \ldots, w_{2n-1}} \right)(0) = 2n - 1. \]

Since \( \varphi_2n = \psi_2n \) and since
\[ J \left( \frac{\varphi_{2n} \circ \tilde{F} \circ \phi}{w_1, \ldots, w_{2n-1}} \right)(0) = J \left( \frac{\psi_{2n} \circ \tilde{F} \circ \phi}{w_1, \ldots, w_{2n-1}} \right)(0) = (0, \ldots, 0), \]
we have
\[ \text{rank } J \left( \frac{u_1, \ldots, u_{2n-1}}{w_1, \ldots, w_{2n-1}} \right)(0) = 2n - 1. \]

Thus \( (u_1, \ldots, u_{2n-1}, u_{2n} = w_{2n}) \) is a coordinate system.

Now, from (2.8) and (2.9), we have
\[ \varphi_i \circ \tilde{F} \circ \phi = u_i, \quad i = 1, 2, \ldots, 2n - 1, \]
\[ \varphi_{2n} \circ \tilde{F} \circ \phi = u_{2n}^{k+1} + \sum_{i=1}^{k-1} w_i u_{2n}^{k-i}. \quad (2.10) \]

Setting \( a_i(u) = w_i \), we obtain (2.7). This completes the proof of Theorem 2.7. ■

**Corollary 2.8** (Symplectomorphic normal form of folds). Let \( \tilde{F} : (U, 0) \to (\mathbb{R}^{2n}, 0) \) be a fold singularity, i.e. an \( A_1 \) type singularity. Then \( \tilde{F} \) is symplectomorphic to the normal
form of the fold:
\[(u_1, \ldots, u_{2n}) \mapsto (u_1, \ldots, u_{2n-1}, u_{2n}^2).\]  \hspace{1cm} (2.11)

Thus the fold type singularities have only one symplectomorphic type.

For \(\bar{F} = (f, g)\) with \(\text{corank} J(\bar{F})(0,0) = k \geq 2\) we get that \(\bar{F}\) can be reduced by symplectomorphic equivalence to the form
\[
\begin{align*}
    f_i(u, v) &= u_i, & i &= 1, \ldots, n - k_1, \\
g_i(u, v) &= v_i, & i &= 1, \ldots, n - k_2, & k_1 + k_2 &= k, k_1 \geq k_2, \\
    \frac{\partial f_i}{\partial u_j}(0,0) &= 0, & n - k_1 < i, j &\leq n, \\
    \frac{\partial f_i}{\partial v_j}(0,0) &= 0, & n - k_1 < i \leq n, n - k_2 < j &\leq n, \\
    \frac{\partial g_i}{\partial u_j}(0,0) &= 0, & n - k_2 < i \leq n, n - k_1 < j &\leq n, \\
    \frac{\partial g_i}{\partial v_j}(0,0) &= 0, & n - k_2 < i, j &\leq n,
\end{align*}
\hspace{1cm} (2.12)
\]

i.e. up to a smooth coordinate change in \(U\) and a symplectomorphic coordinate transformation of \((\mathbb{R}^{2n}, \omega)\).

Below we will consider the special but representative case when \(\bar{F}\) can be reduced by symplectomorphic equivalence to the form
\[
\begin{align*}
    f_i(u, v) &= u_i, & i &= 1, \ldots, n, \\
g_i(u, v) &= v_i, & i &= 1, \ldots, n - k, \\
    \frac{\partial g_i}{\partial v_j}(0,0) &= 0, & n - k < i, j &\leq n.
\end{align*}
\hspace{1cm} (2.13)
\]

with \(k^2 \leq 2n\). The corresponding Jacobi matrix of \(\bar{F}\) can be written in the form
\[
\begin{pmatrix}
    \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\
    \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}
\end{pmatrix}
= \begin{pmatrix}
    I_n & O & O \\
    O & I_{n-k} & O \\
    \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial v_1} & \frac{\partial g_1}{\partial v_m}
\end{pmatrix}
\hspace{1cm} (2.14)
\]

and in blocks
\[
\begin{pmatrix}
    \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\
    \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}
\end{pmatrix}
= \begin{pmatrix}
    I_n & O & O \\
    O & I_{n-k} & O \\
    C & D_1 & D_2
\end{pmatrix}
\hspace{1cm} (2.15)
\]

In this case we can prove a more general version of Theorem 2.5.

**Theorem 2.9.** Let \(F = (f, g, \dot{f}, \dot{g}) : (\mathbb{R}^{2n}, 0) \to T^*\mathbb{R}^{2n}\) be a smooth map-germ such that \(\text{corank} J(\bar{F})(0,0) = k \geq 2\) and \(\bar{F}\) has the form (2.13). Then \(F\) is isotropic if and only if there exists a smooth function \(h\) on \(U\) such that
\[
\begin{pmatrix}
    \frac{\partial h}{\partial u} \\
    \frac{\partial h}{\partial v}
\end{pmatrix}
= \begin{pmatrix}
    \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\
    \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}
\end{pmatrix}
\begin{pmatrix}
    O & -I_n \\
    I_n & O
\end{pmatrix}
\begin{pmatrix}
    \dot{f} \\
    \dot{g}
\end{pmatrix}
\hspace{1cm} (2.16)
\]
which is equivalent to the condition that the component functions of the k-vector given by

\[
\begin{pmatrix}
\frac{\partial h}{\partial v_{n-k+1}} \\
\vdots \\
\frac{\partial h}{\partial v_n}
\end{pmatrix} = \frac{t}{\tilde{D}_2}
\begin{pmatrix}
\frac{\partial h}{\partial v_{n-k+1}} \\
\vdots \\
\frac{\partial h}{\partial v_n}
\end{pmatrix}
\tag{2.17}
\]

belong to \(\langle \det J(f, g) \rangle = \langle \Delta_F \rangle\),

\[
\frac{\partial h}{\partial v_i} \in \langle \det D_2 \rangle = \langle \det J(f, g) \rangle = \langle \Delta_F \rangle, \quad n-k < i \leq n,
\tag{2.18}
\]

where \(\tilde{D}_2\) is the cofactor matrix of \(D_2\).

**Proof.** By matrix calculations we have

\[
\begin{pmatrix}
\dot{f} \\
\dot{g}
\end{pmatrix} = \begin{pmatrix}
O & I_n \\
-I_n & O
\end{pmatrix}^t \begin{pmatrix}
I_n & O & O \\
O & I_{n-k} & O \\
C & D_1 & D_2
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{\partial h}{\partial v} \\
\frac{\partial h}{\partial v_k}
\end{pmatrix}
\tag{2.19}
\]

Thus we have

\[
\begin{pmatrix}
\dot{f}_1 \\
\vdots \\
\dot{f}_{n-k} \\
\dot{f}_{n-k+1} \\
\vdots \\
\dot{f}_n
\end{pmatrix} = \begin{pmatrix}
\frac{\partial h}{\partial v_1} \\
\vdots \\
\frac{\partial h}{\partial v_{n-k}} \\
\frac{\partial h}{\partial v_{n-k+1}} \\
\vdots \\
\frac{\partial h}{\partial v_n}
\end{pmatrix} - t D_1 \begin{pmatrix}
\frac{\partial h}{\partial v_{n-k+1}} \\
\vdots \\
\frac{\partial h}{\partial v_n}
\end{pmatrix},
\tag{2.20}
\]

\[
\begin{pmatrix}
\dot{g}_1 \\
\vdots \\
\dot{g}_n
\end{pmatrix} = - \begin{pmatrix}
\frac{\partial h}{\partial u_1} \\
\vdots \\
\frac{\partial h}{\partial u_n}
\end{pmatrix} + t C \begin{pmatrix}
\frac{\partial h}{\partial v_{n-k+1}} \\
\vdots \\
\frac{\partial h}{\partial v_n}
\end{pmatrix}.
\tag{2.21}
\]

Since the map \((\dot{f}, \dot{g})\) is smooth, all the functions on the right-hand side must be smooth, which holds if and only if

\[
\begin{pmatrix}
\frac{\partial h}{\partial v_{n-k+1}} \\
\vdots \\
\frac{\partial h}{\partial v_n}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial g_{n-k+1}}{\partial v_{n-k+1}} & \cdots & \frac{\partial g_{n-k+1}}{\partial v_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_n}{\partial v_{n-k+1}} & \cdots & \frac{\partial g_n}{\partial v_n}
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{\partial h}{\partial v_{n-k+1}} \\
\vdots \\
\frac{\partial h}{\partial v_n}
\end{pmatrix}
\tag{2.22}
\]

is smooth.

For a square matrix \(A\) of size \(k\), let \(\tilde{A}\) denote the cofactor matrix of \(A\), i.e. the matrix whose \((i, j)\) entry is the cofactor of the \((j, i)\) entry of \(A\). Then we have

\[
\tilde{A} A = A \tilde{A} = \det A \mathbb{I}_k.
\]
Then the formula (2.23) is equal to
\[
\frac{1}{\det D_2} t\tilde{D}_2 \left( \begin{array}{c} \frac{\partial h}{\partial v_{n-k+1}} \\ \vdots \\ \frac{\partial h}{\partial v_n} \end{array} \right) .
\]

Let us denote the \( i-n+k \)-th component of \( t\tilde{D}_2 (\frac{\partial h}{\partial v_{n-k+1}}, \ldots, \frac{\partial h}{\partial v_n}) \) by \( \tilde{h}_i \) (for \( n-k+1 \leq i \leq n \)),
\[
t\tilde{D}_2 \left( \begin{array}{c} \frac{\partial h}{\partial v_{n-k+1}} \\ \vdots \\ \frac{\partial h}{\partial v_n} \end{array} \right) = \left( \begin{array}{c} \tilde{h}_1 \\ \vdots \\ \tilde{h}_n \end{array} \right) .
\]

Thus the right-hand side of (2.19), as a result of (2.20)–(2.22), is smooth if and only if
\[
\frac{\partial h}{\partial v_i} \in \langle \det D_2 \rangle = \langle \det J(f,g) \rangle = \langle \Delta_{\tilde{F}} \rangle, \quad n-k < i \leq n. \quad \blacksquare \quad (2.24)
\]

3. Transversality of isotropic mappings. We find a condition on an isotropic map germ \( F \) ensuring that \( j^1 F \) is transversal to the corank 1 stratum in the isotropic 1-jet space of mappings when \( \tilde{F} \) has corank 1 at the origin. The case of corank \( k \) singularity of \( \tilde{F} \) when \( k > 2 \) will also be considered. But the case \( k = 2 \) we leave to the forthcoming paper.

Let us identify the space of 1-jets \( J^1(\mathbb{R}^{2n}, T\mathbb{R}^{2n}) \) with \( \mathbb{R}^{2n} \times T\mathbb{R}^{2n} \times M(4n, 2n) \), where \( M(4n, 2n) \) is the set of \( 4n \times 2n \) matrices,
\[
J^1(\mathbb{R}^{2n}, T\mathbb{R}^{2n}) = \mathbb{R}^{2n} \times T\mathbb{R}^{2n} \times M(4n, 2n) = \mathbb{R}^{2n} \times T\mathbb{R}^{2n} \times \mathbb{R}^{2n \times 4n}.
\]

Let \( (a, b, c, d) = (a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots, d_n) \) denote the canonical coordinates of \( \mathbb{R}^{4n} = T\mathbb{R}^{2n} \) endowed with the symplectic structure
\[
\omega = \sum_{i=1}^{n} (dd_i \wedge da_i - dc_i \wedge db_i).
\]

Let \( A = ^t(a_{ij}, b_{ij}, c_{ij}, d_{ij}) \in M(4n, 2n), 1 \leq i \leq n, 1 \leq j \leq 2n \) and \( ^t(a_j, b_j, c_j, d_j) \) denotes the \( j \)-th column of \( A, 1 \leq j \leq 2n \). Then
\[
A \quad \text{is isotropic if} \quad \omega(^t(a_i, b_i, c_i, d_i), ^t(a_j, b_j, c_j, d_j)) = 0,
\]
for all \( 1 \leq i, j \leq 2n \), where
\[
\omega(^t(a_i, b_i, c_i, d_i), ^t(a_j, b_j, c_j, d_j)) := \sum_{k=1}^{n} ((a_{kj}d_{ki} - a_{ki}d_{kj}) - (b_{kj}c_{ki} - b_{ki}c_{kj})).
\]

We define the subsets
\[
I = I(4n, 2n) = \{ A \in M(4n, 2n) : A \text{ is isotropic} \},
\]
\[
I_k = I_k(4n, 2n) = \{ A \in I(4n, 2n) : \text{corank} A = k \}.
\]

By \( \overline{I_k} \) we denote the topological closure of \( I_k \),
\[
\overline{I_k} = I_k \cup I_{k+1} \cup \cdots \cup I_{2n}.
\]
$I(4n, 2n)$ has singularities along $\mathcal{T}_2$ and $I(4n, 2n) - \mathcal{T}_2$ is a codimension $n(2n - 1)$ smooth submanifold of $M(2n, 4n)$. Let $\mathcal{J}^1_1(\mathbb{R}^{2n}, T\mathbb{R}^{2n})$ denote the space of 1-jets of isotropic maps with corank at most 1, i.e.

$$\mathcal{J}^1_1(\mathbb{R}^{2n}, T\mathbb{R}^{2n}) = \mathbb{R}^{2n} \times T\mathbb{R}^{2n} \times (I(4n, 2n) - \mathcal{T}_2).$$

Let $F : \mathbb{R}^{2n} \supset U \rightarrow T\mathbb{R}^{2n}$ be a smooth isotropic map such that $F$ and $\bar{F} = \pi \circ F$ have a corank 1 singularity at $(0, 0) \in \mathbb{R}^{2n}$ and $\bar{F}$ has the form (2.3) in local coordinates $(u, v)$. Consider the 1-jet extension $j^1 F : U \rightarrow \mathcal{J}^1_1(\mathbb{R}^{2n}, T\mathbb{R}^{2n})$ given by $j^1 F(u, v) = (u, v, F(u, v), J(F)(u, v)).$ Thus, $j^1 F$ is transversal to the corank 1 stratum $\mathbb{R}^{2n} \times T\mathbb{R}^{2n} \times I_1$ if and only if $J(F) : U \rightarrow I(4n, 2n) - \mathcal{T}_2$ is transversal to $I_1$. Now we seek a genericity condition for $F$ in order that $J(F)$ be transversal to $I_1$.

If $F$ is isotropic, then by Theorem 2.5 for its generating function $h$ we have

$$\frac{\partial h}{\partial v_n}(u, v) = \alpha(u, v)\Delta_F$$

for some smooth function $\alpha(u, v)$.

**Proposition 3.1.** The corank of $J(F)(u, v)$ is 1 if and only if

$$\Delta_F(u, v) = 0 \quad \text{and} \quad \frac{\partial \alpha}{\partial v_n}(u, v) = 0.$$

**Proof.** For the purpose of this proof we use the notation $(w_1, \ldots, w_{2n}) = (u_1, \ldots, u_n, v_1, \ldots, v_n)$. Since the corank of $J(F)(0, 0)$ is 1, corank$J(F)(u, v) \leq 1$. Thus in order that corank$J(F)(u, v) = 1$, it is necessary that $\Delta_F(u, v) = 0$. So, under the assumption that $\Delta_F(u, v) = 0$, we prove that the corank of $J(F)(u, v)$ is one if and only if $\frac{\partial \alpha}{\partial w_{2n}}(u, v) = 0$.

If corank$J(F)(u, v) = 1$ then we have

$$0 = \frac{\partial f_n}{\partial w_{2n}}(u, v) = \frac{\partial \alpha}{\partial w_{2n}}(u, v).$$

Thus we have $\frac{\partial \alpha}{\partial w_{2n}}(u, v) = 0$.

Now suppose that $\frac{\partial \alpha}{\partial w_{2n}}(u, v) = 0$. From (3.1) we can write

$$\frac{\partial f_i}{\partial w_j} = \frac{\partial^2 h}{\partial w_{n+i}\partial w_j} - \frac{\partial^2 g_n}{\partial w_{n+i}\partial w_j} \alpha - \frac{\partial g_n}{\partial w_{n+i}} \frac{\partial \alpha}{\partial w_j},$$

$$\frac{\partial f_n}{\partial w_j} = \frac{\partial \alpha}{\partial w_j},$$

$$\frac{\partial g_i}{\partial w_j} = -\frac{\partial^2 h}{\partial w_i\partial w_j} + \frac{\partial^2 g_n}{\partial w_i\partial w_j} \alpha + \frac{\partial g_n}{\partial w_i} \frac{\partial \alpha}{\partial w_j}, \quad j = 1, \ldots, 2n.$$

Thus the $j$-th column $a_j$ of the Jacobian matrix $J(F)$ for $j < 2n$ can be written in the form

$$a_j = \left(0, \ldots, 1, 0, \ldots, 0, \frac{\partial g_n}{\partial w_j}, \ldots, \frac{\partial^2 h}{\partial w_{n+i}\partial w_j} - \frac{\partial^2 g_n}{\partial w_{n+i}\partial w_j} \alpha - \frac{\partial g_n}{\partial w_{n+i}} \frac{\partial \alpha}{\partial w_j}, \ldots, \frac{\partial \alpha}{\partial w_j}, \ldots, -\frac{\partial^2 h}{\partial w_i\partial w_j} + \frac{\partial^2 g_n}{\partial w_i\partial w_j} \alpha + \frac{\partial g_n}{\partial w_i} \frac{\partial \alpha}{\partial w_j}, \ldots \right)$$
And the $2n$-th column is
\[
a_{2n} = \left(0, \ldots, 0, \Delta_F, \ldots, \frac{\partial^2 h}{\partial w_{n+i} \partial w_{2n}}, \frac{\partial^2 h}{\partial w_{n+i} \partial w_{2n}} - \frac{\partial^2 g_n}{\partial w_{n+i} \partial w_{2n}} \alpha - \frac{\partial g_n}{\partial w_{n+i} \partial w_{2n}}, \ldots \right).
\]
Since $F$ is isotropic, for $j \leq n$, we have
\[
0 = \dot{\omega}(a_j, a_{2n}) = \frac{\partial^2 h}{\partial w_j \partial w_{2n}} - \frac{\partial^2 g_n}{\partial w_j \partial w_{2n}} \alpha - \frac{\partial g_n}{\partial w_j \partial w_{2n}} + \frac{\partial g_n}{\partial w_j \partial w_{2n}} - \Delta_F \frac{\partial \alpha}{\partial w_j},
\]
and
\[
0 = \dot{\omega}(a_{n+j}, a_{2n}) = \frac{\partial^2 h}{\partial w_{n+j} \partial w_{2n}} - \frac{\partial^2 g_n}{\partial w_{n+j} \partial w_{2n}} \alpha - \frac{\partial g_n}{\partial w_{n+j} \partial w_{2n}} + \frac{\partial g_n}{\partial w_{n+j} \partial w_{2n}} - \Delta_F \frac{\partial \alpha}{\partial w_{n+j}}.
\]
Thus in both cases, we have
\[
0 = \dot{\omega}(a_j, a_{2n}) = \frac{\partial^2 h}{\partial w_j \partial w_{2n}} - \frac{\partial^2 g_n}{\partial w_j \partial w_{2n}} \alpha - \Delta_F \frac{\partial \alpha}{\partial w_j}.
\]
Since $\Delta_F(u, v) = 0$, we have
\[
\frac{\partial^2 h}{\partial w_{n+j} \partial w_{2n}}(u, v) - \frac{\partial^2 g_n}{\partial w_{n+j} \partial w_{2n}}(u, v)\alpha(u, v) = 0.
\]
Now, since we assumed that $\frac{\partial \alpha}{\partial w_{2n}}(u, v) = 0$, from (3.2) and (3.4), for all $i$, we have
\[
\frac{\partial \dot{f}_i}{\partial w_{2n}}(u, v) = 0, \quad \frac{\partial \dot{g}_i}{\partial w_{2n}}(u, v) = 0.
\]
Thus $J(F)(u, v)$ has corank 1. ■

We can write
\[
(jF)^{-1}(I_1) = \left\{(u, v) \in U : \Delta_F(u, v) = 0 \quad \text{and} \quad \frac{\partial \alpha}{\partial w_{2n}}(u, v) = 0 \right\}
\]
and by Proposition 3.1 we have

**Proposition 3.2.** Let $F : (U, 0) \to T\mathbb{R}^{2n}$ be an isotropic map-germ generated by a smooth function-germ $h$ satisfying (3.1) such that corank of $JF(0, 0)$ is equal to 1. Then $j^1F : (U, 0) \to I(4n, 2n)$ is transversal to $I_1$ at $(0, 0)$ if and only if
\[
\text{rank} J\left(\Delta_F, \frac{\partial \alpha}{\partial v_n}\right)(0, 0) = 2.
\]

Now we compare a generic property of a smooth Lagrangian submanifold $L \subset T\mathbb{R}^{2n}$ generated by a versal Morse family germ with a corresponding one obtained for an
isotropic mapping $F$. Let $\pi|_L : L \to \mathbb{R}^{2n}$ denote the restriction of $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ to $L$ and

$$\Sigma^i(\pi|_L) = \{ p \in L : \text{corank}(\pi|_L)_p = i \}.$$ 

It is well known that

1) the codimension of $\Sigma^1(\pi|_L)$ in $L$ is 1 and

2) the codimension of $\Sigma^i(\pi|_L)$ in $L$ is $i(i + 1)/2 \geq 3$ if $i \geq 2$.

On the other hand for a map-germ $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ such that $j^1\bar{F}$ is transversal to the corank $k$ stratum in the jet space for all $k = 0, \ldots, 2n$ we have a corresponding property,

1) the codimension of $\Sigma^1(\bar{F})$ in $U$ is 1 and

2) the codimension of $\Sigma^i(\bar{F})$ in $U$ is $i^2 \geq 4$ if $i \geq 2$.

Let us denote

$$\Sigma^k(\mathbb{R}^{2n}, \mathbb{R}^{2n}) = \{ \sigma \in J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n}) : \text{corank}\sigma = k \}.$$ 

Then we have

**Lemma 3.3.** Let $L \subset T\mathbb{R}^{2n}$ be a Lagrangian submanifold. Let $p \in L$ and suppose that the corank of the differential $d(\pi|_L)_p : T_pL \to T_{\pi(p)}\mathbb{R}^{2n}$ is $k \geq 2$. Then $j^1(\pi|_L) : L \to J^1(L, \mathbb{R}^{2n})$ is not transversal to $\Sigma^k(L, \mathbb{R}^{2n}) \subset J^1(L, \mathbb{R}^{2n})$ at $p$.

And using Lemma 3.3 we get

**Theorem 3.4.** Suppose that $n \geq 2$ and $k \geq 2$. Let $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ be a smooth map-germ such that $j^1\bar{F}(0, 0) \in \Sigma^k$ and that $j^1\bar{F}$ is transversal to $\Sigma^k(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. Let $F : (U, 0) \to T\mathbb{R}^{2n}$ be a corank 1 isotropic map-germ along $\bar{F}$. Then $F$ is neither a Lagrangian immersion nor a Lagrangian stable isotropic map-germ.

In [8] G. Ishikawa classified Lagrangian stable isotropic map-germs of corank 1 and named them open Whitney umbrellas (cf. [7, 12]). In our context, his theorem can be stated as follows.

**Theorem 3.5 (Ishikawa [8]).** Let $F : (U, 0) \to T\mathbb{R}^{2n}$ be a Lagrangian stable isotropic map-germ of corank 1. Then $F$ is Lagrangian equivalent to one of the following normal forms $F_{2n,k} = (f, g, \dot{f}, \dot{g}) : (U, 0) \to T\mathbb{R}^{2n}$ defined by

\[
\begin{align*}
    f_1(u, v) &= u_i = w_i, \\
    g_1(u, v) &= v_i = w_{n+i}, \quad i = 1, \ldots, n-1, \\
    g_n(u, v) &= \frac{v_{n+k}^{k+1}}{(k+1)!} + u_1\frac{v_{n+k}^{k-1}}{(k-1)!} + \cdots + u_{k-1}v_n, \\
    \dot{g}_n(u, v) &= w_k\frac{v_{n+k}^k}{k!} + \cdots + w_{2k-1}v_n,
\end{align*}
\]
\[f_i(u, v) = \frac{-1}{i!} \left( \frac{u_{2n+i+1}^{k+i+1}}{(k+i+1)k!} + \frac{w_1u_{2n+i+1}^{k+i-1}}{(k+i-1)(k-2)!} + \cdots + \frac{w_{k-1}u_{2n+i+1}}{i+1} \right)\]

for \(i\) with \(k \leq n + i \leq 2k - 1\),

\[\hat{f}_i(u, v) = 0\]

for \(i\) with \(2k \leq n + i\),

\[\hat{g}_i(u, v) = \frac{-1}{(k-i)!} \left\{ \frac{w_kw_{2n}^{2k-i}}{(2k-i)(k-1)!} + \frac{w_{k+1}u_{2n}^{2k-i-1}}{(2k-i-1)(k-2)!} + \cdots + \frac{w_{k-1}u_{2n}^{k-i+1}}{i+1} \right\}\]

for \(i \leq k - 1\),

\[\hat{g}_i(u, v) = \frac{-1}{i!} \left( \frac{u_{2n}^{k+i+1}}{(k+i+1)k!} + \frac{w_1u_{2n}^{k+i-1}}{(k+i-1)(k-2)!} + \cdots + \frac{w_{k-1}u_{2n}^{i+1}}{i+1} \right)\]

for \(k \leq i \leq 2k - 1\),

\[\hat{g}_i(u, v) = 0\]

for \(i \geq 2k\),

where \((w_1, \ldots, w_{2n}) = (u_1, \ldots, u_n, v_1, \ldots, v_n)\).

**Proof of Theorem 3.4.** First we show that if \(j^1\bar{F}(0, 0) \in \Sigma^k\) and \(j^1\bar{F}: U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})\) is transversal to \(\Sigma^k\) at the origin \((0, 0) \in \mathbb{R}^{2n}\) and \(F: (U, 0) \to T\mathbb{R}^{2n}\) is an isotropic map-germ such that \(\pi \circ F = \bar{F}\), then the origin \((0, 0)\) is a singular point of \(F\). Indeed, suppose that the origin \((0, 0)\) is a regular point of \(F\). Then \(F: (U, 0) \to T\mathbb{R}^{2n}\) is a Lagrangian immersion-germ, choosing \(U\) small enough if necessary. Set \(L = F(U)\). Then \(L\) is a Lagrangian submanifold of \(T\mathbb{R}^{2n}\). We see that \(\pi \circ F = \bar{F}\) and \(j^1(\pi|_L): L \to J^1(L, \mathbb{R}^{2n})\) is transversal to \(\Sigma^k \subset J^1(L, \mathbb{R}^{2n})\) at \(F(0, 0)\) if and only if \(j^1\bar{F}: U \to J^1(U, \mathbb{R}^{2n})\) is transversal to \(\Sigma^k \subset J^1(U, \mathbb{R}^{2n})\) at \((0, 0)\). But from Lemma 3.3 we know that \(j^1(\pi|_L): L \to J^1(L, \mathbb{R}^{2n})\) is never transversal to \(\Sigma^k \subset J^1(L, \mathbb{R}^{2n})\) at \(F(0, 0)\). So we have got a contradiction, thus the origin is a singular point of \(F\) and \(F\) is not a Lagrangian immersion.

The fact that \(F\) is not a Lagrangian stable isotropic map-germ of corank 1 can be seen as follows. For some symplectomorphism \(\Psi: T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}\), the composed map \(\pi \circ \Psi \circ F_{2n,\ell}\) of \(\pi, \Psi\) and an open Whitney umbrella \(F_{2n,\ell}\) may have corank \(k\) singular points. However, from Ishikawa’s normal forms, it is easy to see that for none of them, \(j^1\pi \circ \Psi \circ F_{2n,\ell}\) is transversal to \(\Sigma^k(\mathbb{R}^{2n}, \mathbb{R}^{2n})\). Therefore the isotropic map-germ \(F\) in question is not symplectomorphic to any of Ishikawa’s normal forms \(F_{2n,\ell}\) and \(F\) is not Lagrangian stable.

**Remark 3.6.** For \(k = 1\), if the corank of the differential \(d(\pi|_L)_p : T_pL \to T_{\pi(p)}\mathbb{R}^{2n}\) at \(p \in L\) is 1 and \(L\) is generated by a versal Morse family, then \(j^1(\pi|_L): L \to J^1(L, \mathbb{R}^{2n})\) is transversal to \(\Sigma^1 \subset J^1(L, \mathbb{R}^{2n})\) at \(p\).

Now we find the condition for transversality to the corank 1 stratum of isotropic maps with corank of \(J(\bar{F})\) greater than or equal to 2.

As in Section 2, we consider an isotropic map-germ \(F = (f, g, \dot{f}, \dot{g}): (U, 0) \to T\mathbb{R}^{2n}\) with corank \(J(\bar{F})(0, 0) = k \geq 2\) with \(\bar{F} = (f, g)\) of the form (2.12)–(2.15). Let \(h\) be a generating function of \(F\), i.e. it satisfies (2.16). Let
\[ \frac{\partial h}{\partial v_i} \in \langle \text{det} D^2 \rangle = \langle \text{det} J(f, g) \rangle = \langle \Delta_F \rangle, \quad n - k < i \leq n, \]

be the functions given by (2.17). Since \((\dot{f}, \dot{g})\) is given by (2.20)-(2.22), \(j^1 F\) meets the codim 1 stratum if and only if

\[
\text{rank} \left( \frac{\partial}{\partial (v_{n-k+1}, \ldots, v_n)} \left( \frac{\partial h}{\partial u_1}, \ldots, \frac{\partial h}{\partial v_n}, \frac{\partial h}{\partial v_{n-k+1}}, \ldots, \frac{\partial h}{\partial v_n} \right) \right) = k - 1. \quad (3.5)
\]

Thus we have

**Theorem 3.7.** Let \(F, h, \tilde{\partial}h/\partial v_i, n - k < i \leq n\) be as above:

\[
\frac{\tilde{\partial}h}{\partial v_i} \in \langle \Delta_F \rangle, n - k < i \leq n \quad \text{and} \quad \left( \begin{array}{c} \dot{f} \\ \dot{g} \end{array} \right) = \left( \begin{array}{cc} O & I_n \\ -I_n & O \end{array} \right) \left( \begin{array}{cc} \partial f/\partial u_1 & \partial f/\partial v_1 \\ \partial g/\partial u_1 & \partial g/\partial v_1 \end{array} \right)^{-1} \left( \begin{array}{c} \partial h/\partial u_1 \\ \partial h/\partial v_1 \end{array} \right). \quad (3.6)
\]

1) The corank of \(j^1 F\) at \((0, 0)\) is 1 if and only if

\[
\text{rank} \left( \begin{array}{cccccccc} \frac{\partial^2 h}{\partial u_1 \partial v_{n-k+1}} & \cdots & \frac{\partial^2 h}{\partial u_1 \partial v_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 h}{\partial v_n \partial v_{n-k+1}} & \cdots & \frac{\partial^2 h}{\partial v_n \partial v_n} \end{array} \right) (0, 0) = k - 1. \quad (3.6)
\]

2) The jet extension \(j^1 F : U \to J^1_1(\mathbb{R}^{2n}, T\mathbb{R}^{2n})\) is transversal to the corank 1 stratum if and only if

\[
\text{rank} \left( J(k \times k \text{ minors of the matrix } (3.6)) \right) = 2, \quad (3.7)
\]

where \(J(k \times k \text{ minors of the matrix } (3.6))\) is the Jacobian matrix at \((0, 0)\) of the \(k \times k\) minors of the matrix (3.6) with respect to the variables \(u_1, \ldots, u_n, v_1, \ldots, v_n\).

**Proof.** 1) follows from (3.5). 2) is also straightforward by the fact that the corank 1 stratum is defined by the minors of the matrix (3.6) and the codimension of the corank 1 stratum is 2. 


References


