LECTURE 5

Bell numbers (continued)

Theorem.

\[ B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i \quad (B_0 \text{ is assumed to be 1}). \]

Proof

We will split \( \Pi(X) \) into disjoint subsets. With each subset \( T \) of \( X=[n+1] \) containing \( n+1 \) we associate the set of those partitions of \( X \) who contain \( T \) as one of their blocks. For each such \( T \) there are exactly \( B_{|X\setminus T|} \) partitions of \( X \) who contain \( T \) as a block, \( X\setminus T \) being a subset of \( \{1,2, \ldots ,n\} \). Since for each \( i=0,1,2, \ldots ,n \) we have \( \binom{n}{i} \) \( i \)-element subsets of the type \( X\setminus T \), each can be partitioned in \( B_i \) ways and those partitions, together with the set, \( T \) form partitions of \( X \), we have the total of

\[ \sum_{i=0}^{n} \binom{n}{i} B_i \] partitions of \( X \). \( \Box \)

Here come the first 10 Bell numbers: \( B_0=1, B_1=1, B_2=2, B_3=5, B_4=15, B_5=52, B_6=203, B_7=877, B_8=4140, B_9=21147, B_{10}=115975. \)

Enumeration of partitions of a set

An easy recursive algorithm enumerating all partitions of \([n]\) can be based on this simple observation. If \( \pi = \{B_1, B_2, \ldots , B_k\} \) is a partition of \([n-1]\) then \( \pi^* = \{\{\{n\}\}, B_1, B_2, \ldots , B_k\}, \{B_1, B_2, \ldots , B_k\}, \ldots , \{B_1, \ldots , \{n\}\}\} \) is a set of partitions of \([n]\), and if we have two different partitions \( \pi \) and \( \sigma \) of \([n-1]\) then \( \pi^* \cap \sigma^* = \emptyset \).

This means that to enumerate all partitions of \([n]\) we list all partitions of \([n-1]\) and blow up each partition to its asterisked child. Let us do this for \( n=4 \). We start with \([1]\), and of course there is just one partition \( \{\{1\}\} \). The asterisk operation applied to this results in \( a^* = \{\{1,2\}\} \) and \( b^* = \{\{1\}\} \). Next we get \( a^* = \{\{1,2,3\}\}, \{\{1\}\}, \{\{2\}\}, \{\{3\}\} \) and \( b^* = \{\{1\}\}, \{\{2\}\}, \{\{3\}\} \) to the total of 5 partitions. These give us:

\[
\begin{align*}
\{\{1,2,3\}\}^* &= \{\{1,2,3,4\}\}, \{\{1,2,3\}\}, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}\}
\{\{1,2\}\}^* &= \{\{1,2,3\}\}, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}\}
\{\{1,3\}\}^* &= \{\{1,2,3\}\}, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}\}
\{\{1\}\}^* &= \{\{1\}\}, \{\{2\}\}, \{\{3\}\}\}
\end{align*}
\]

which brings the total to \( 2+3+3+3+4 = 15 \) partitions, as predicted by adding all entries in row four of our Stirling triangle.

Partitions of a number

Definition.

Given a natural number \( n \), every sequences of positive integers \( (a_1, a_2, \ldots , a_k) \) such that \( a_1+a_2+ \ldots +a_k = n \) is referred to as an ordered partition of \( n \) into \( k \) summands or parts.

One can imagine an ordered partition of \( n \) as a row of \( n \) ones separated into \( k \) smaller rows by \( k-1 \) zeroes. The difference between this and an \( n \)-combination with repetitions of a \( k \)-element set is that
we disallow empty parts. This means that no zero is allowed as either the first or the last element of the sequence, and there are no two consecutive zeroes. Hence the number of ordered partitions equals the number of ways we can place \( k - 1 \) zeroes in \( n - 1 \) spaces between ones, i.e. \( \binom{n-1}{k-1} \). The problem of counting unordered partitions turns out to be much more complicated. If we want to disregard the ordering of parts, we can choose a particular ordering and consider it a representation of all ordered partitions with the same parts. Hence, we can define unordered partitions as follows.

**Definition.**
Given a natural number \( n \), every non-increasing sequence of positive integers \( (a_1, a_2, \ldots, a_k) \) such that, \( a_1 + a_2 + \ldots + a_k = n \) is referred to as an unordered partition of \( n \) into \( k \) summands, or simply a partition.

Thus, \( a_1 \) is the largest, and \( a_k \) the smallest element of the sequence.

We will denote the number of partitions of \( n \) by \( P(n) \). The number of partitions of \( n \) into \( k \) summands will be denoted by \( P(n, k) \).

**Theorem.**

1. \( P(n) = \sum_{i=1}^{n} P(n, k) \)
2. \( P(n, 1) = P(n, n) = 1 \)
3. \( P(n, 2) = \frac{n}{2} P(n, n-1) = 1. \Box \)

**Example.**
Number 4 can be partitioned into 4, 3+1, 2+2, 2+1+1 and 1+1+1+1. This means \( P(4,1)=1, P(4,2)=2, P(4,3)=1 \) and \( P(4,4)=1 \).

A very convenient and surprisingly effective way of representing a partition \( (a_1, a_2, \ldots, a_k) \) of \( n \) is the Ferrer’s diagram. It is simply a \( k \times a_1 \) 0-1 matrix, whose row \( i \) consists of \( a_i \) ones, followed by \( a_1 - a_i \) zeroes, \( i=1,2,\ldots,k \). For example, the diagrams

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

represent the 5+4+2+1+1 partition of 13, 7+5+3+2 partition of 17, 1+1+1+1+1 partition of 5, and 7+1+1+1+1 partition of 11.

**Definition.**
If a matrix \( A \) is the Ferrer’s diagram of a partition \( \pi \) of \( n \) then \( A^T \) (A transpose) is the Ferrer’s diagram of another partition of \( n \), called the conjugate of \( \pi \).

**Theorem.**
\( P(n,k) \) is equal to the number of partitions of \( n \) with the largest summand equal to \( k \).

**Proof.**
The conjugate of a partition of \( n \) into \( k \) summands \( a_1, a_2, \ldots, a_k \) is a partition of \( n \) into \( a_1 \) summands, with the largest summand equal to \( k \). Moreover, conjugation is clearly a bijection, mapping the set of
all partitions of \( n \) into \( k \) summands into the set of all partitions of \( n \) with the largest summand equal to \( k \). □

**Theorem.**
If \( n > k \) and \( p_s(t) \) denotes the number of partitions of \( t \) into no more than \( s \) summands then
\[
P(n,k) = p_k(n - k).
\]

**Proof.**
The left hand side is the number of all Ferrer’s diagrams with \( k \) rows and \( n \) ones. The leading column in each diagram consists of \( k \) ones. From each diagram let us remove the leading column and resulting all-zero rows, if any. In this way to each Ferrer’s diagram with \( k \) rows and \( n \) ones we assign different Ferrer’s diagram with at most \( k \) rows and exactly \( n-k \) ones. Since adding a column consisting of \( k \) ones to each Ferrer’s diagram with at most \( k \) rows and exactly \( n-k \) ones is clearly the inverse assignment we are done. □

**Example.**

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{array}
\]

The assignment leads from a 5-partition of 15 into 6+4+3+1+1 to a 3-partition of 10 into 5+3+2.

**Corollary.**
If \( n > k \) then
\[
P(n,k) = \sum_{i=1}^{k} P(n-k,i).
\]

**Corollary.**
If \( n > k > 1 \) then
\[
p_k(n) = p_{k-1}(n) + p_k(n-k)
\]

**Proof.**
We separate partitions of \( n \) with at most \( k \) parts into two disjoint subsets: partitions with less than \( k \) parts, i.e. at most \( k-1 \) parts and those into exactly \( k \) parts. The size of the first part, by definition is \( p_{k-1}(n) \), and the size of the second part is \( P(n,k) = p_k(n-k) \), by the theorem. □

**Example.**
The first corollary can be employed to fill out the matrix containing \( P(n,k) \) in \( n \)-th row and \( k \)-th column. The matrix is a distant cousin of Pascal’s triangle. The recipe for \( P(n,k) \) is “go \( k \) steps up from position \( (n,k+1) \) and sum up all entries on you left. This is the result for \( n \) and \( k \) up to 10:

\[
\begin{array}{cccccccccccc}
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
6 & 1 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
7 & 1 & 3 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\
8 & 1 & 4 & 5 & 4 & 3 & 2 & 1 & 1 & 0 & 0 \\
9 & 1 & 4 & 8 & 7 & 4 & 3 & 2 & 1 & 1 & 0 \\
10 & 1 & 5 & 10 & 9 & 7 & 5 & 3 & 2 & 1 & 1 \\
\end{array}
\]

The matrix allows you to easily calculate \( p_k(n) \).