Finite axiomatization for quasivarieties of digraphs

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PROBLEM

Fragments of first order logic

Hierarchy

- full FOL
- ....
- $\exists\forall$-fragment of FOL
  $$\exists\bar{x}\forall\bar{y}\ \varphi \qquad (\varphi \text{ quantifier-free})$$
- $\forall$-fragment of FOL
  $$\forall\bar{x}\ \varphi \qquad (\varphi \text{ quantifier-free})$$
- quasi-equational (universal-Horn) fragment of FOL
  $$\forall\bar{x}\ [\varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi] \qquad (\varphi_i, \varphi \text{ atomic})$$
- equational fragment of FOL $\qquad \forall\bar{x}\ \varphi \qquad (\varphi \text{ atomic})$

D - (finite) digraph (or structure in general), F - fragment of FOL

$Th_F(\mathbf{D}) = \{\psi \in F \mid \mathbf{D} \text{ satisfies } \psi\}$ - F-theory of **D**.

Finite axiomatization problem

Problem

$\mathbf{D}$ - finite digraph (or relational structure)
F - fragment of FOL
Is $Th_F(\mathbf{D})$ finitely axiomatizable?

Solution - easy parts

- Yes for full FOL (trivial)
- Yes for $\exists\forall$-fragment of FOL (as above)
- Yes for $\forall$-fragment of FOL (easy)
- ??? for quasi-equational fragment of FOL
- Yes for equational fragment of FOL (easy)

Theorem (Grnostaev '85)

There is a 2-element structure $\mathbf{S}$ with a ternary relation for which $Th_{QE}(\mathbf{S})$ is not finitely axiomatizable.

Semantics

Fact

$\mathbf{E}$ satisfies $Th_{QE}(\mathbf{D})$ (for finite $\mathbf{D}$) iff $\mathbf{E}$ is a (induced) substructure of a (direct) power of $\mathbf{D}$

$$\mathbf{E} \in \mathrm{SP}(\mathbf{D})$$

Fact

$\mathbf{E} \in \mathrm{SP}(\mathbf{D})$ iff $\forall (a, b) \in E^2$

- $a \neq b \quad \Rightarrow \quad \exists h \colon \mathbf{E} \to \mathbf{D}$ s.t. $h(a) \neq h(b)$
- (a, b) non-edge $\quad \Rightarrow \quad \exists h \colon \mathbf{E} \to \mathbf{D}$ s.t. $(h(a), (b))$ non-edge

Fact

Finite member problem for $\mathrm{SP}(\mathbf{D})$ is

- NP-complete in general
- in uniform AC^0 when $Th_{QE}(\mathbf{D})$ is finitely axiomatizable

Algebraic inspiration

Tarski's Problem ('60s)

Is there an algorithm which, given an arbitrary finite algebra as input, determines whether its equational theory is finitely axiomatizable?

R. McKenzie's Answer '96

No!

Quasi-Tarski's Problem

Is there an algorithm which, given an arbitrary finite algebra/structure as input, determines whether its quasi-equational theory is finitely axiomatizable?

Answer is unknown

Modification of McKenzie's method for algebras failed.

KNOWN

Simple graphs

Theorem (Nešetřil, Pultr '78)

Let $\mathcal{K}$ be a finite family of finite simple graphs. Then $Th_{QE}(\mathcal{K})$ is finitely axiomatizable iff $\mathsf{SP}(\mathcal{K})$ is among

- $\mathsf{SP}(\emptyset)$
- $\mathsf{SP}(\bullet)$
- $\mathsf{SP}(\bullet \quad \bullet)$
- $\mathsf{SP}(\bullet\!-\!\bullet)$
- $\mathsf{SP}(\bullet\!-\!\bullet\!-\!\bullet)$

Corollary

It is decidable whether $Th_{QE}(\mathbf{G})$, for a simple graph $\mathbf{G}$, is finitely axiomatizable

Anti-equational theory and constraints satisfaction problem

Anti-identities looks like

$$\forall \bar{x}\ [\neg\varphi_1 \vee \cdots \vee \neg\varphi_n] \qquad (\varphi_i \text{ atomic})$$

$$CSP(\mathbf{T}) = \{\mathbf{D} \text{ finite} \mid \exists h\colon \mathbf{D} \to \mathbf{T}\}$$

Fact

$\mathbf{D} \in CSP(\mathbf{T})$ iff $\mathbf{D}$ satisfies $Th_{AE}(\mathbf{T})$

AE-theory and CSP cont.

Therem (Atserias, Larose, Loten, Rossman, Tardif '08)

Let $\mathbf{T}$ be a finite relational structure. TFAE

- $Th_{AE}(\mathbf{T})$ is finitely axiomatizable
- CSP($\mathbf{T}$) is finitely axiomatizable relative to finite structures (membership problem for CSP($\mathbf{M}$) is in uniform AC^0)
- Core$(\mathbf{T})^2$ dismantles to the diagonal.

Corollary

It is decidable whether $Th_{AE}(\mathbf{T})$, for a finite relational structure $\mathbf{T}$, is finitely axiomatizable.

NEW

Critical digraphs

$\mathbf{E}, \mathbf{D}$ - digraphs
$\mathbf{E}$ is $\mathbf{D}$-critical if $\mathbf{E} \notin \mathrm{SP}(\mathbf{D})$ and
$\forall \mathbf{F} \quad \mathbf{F} < \mathbf{E} \quad \Rightarrow \quad \mathbf{F} \in \mathrm{SP}(\mathbf{D})$

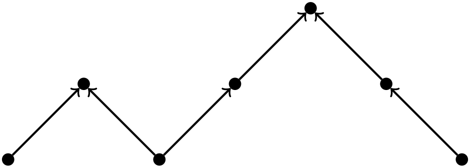
Fact

$Th_{QE}(\mathbf{D})$ is not finitely axiomatizable iff there are arbitrary large $\mathbf{D}$-critical digraphs

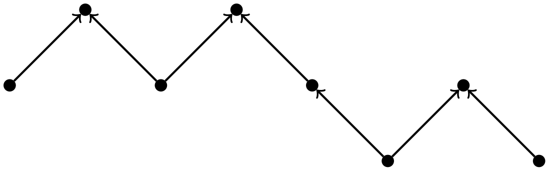
Working Conjecture

There is a computable function f such that for every digraph $\mathbf{D}$, if there is a $\mathbf{D}$-critical structure of cardinality $> f(|D|)$, then there are arbitrary large $\mathbf{D}$-critical structures.

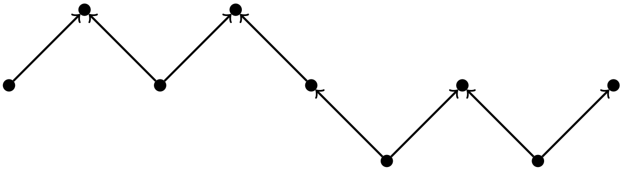
Oriented path looks like



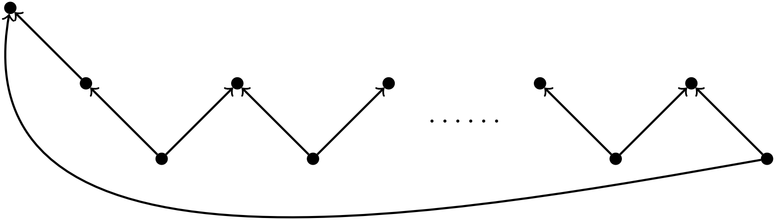
Example for finite axiomatizability



Example for non-finite axiomatizability

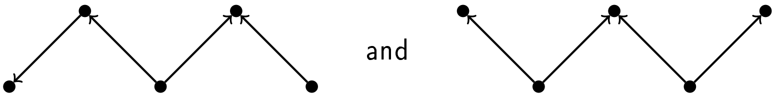


Critical digraphs



Oriented paths with zigzags

By zigzags we mean



Theorem (Jackson, Kowalski, S. '13+)

Let **P** be an oriented path with a zigzag (as a subdigraph). If there is a **P**-critical digraph of cardinality at least $8|P| + 13$, then there exists an arbitrary large **P**-critical digraph.

Corollary

It is decidable whether $Th_{QE}(\mathbf{P})$, for a finite oriented path with a zigzag **P**, is finitely axiomatizable.

TIME FOR COOKIES