A new proof of Pigozzi theorem

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Outline

1. Finite basis of quasivarieties

2. A proof of Pigozzi theorem
Quasivarieties

Definition

A quasivariety is a class of algebras closed under $S$, $P$ and $P_U$. 
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If $\mathcal{F}$ is a finite family of finite algebras then the smallest quasivariety containing $\mathcal{F}$ equals to $\text{SP}(\mathcal{F})$. 

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Fact

A Class is a quasivariety iff it is definable by quasi-identities, that is universal formulae of the form

$$\left( \forall \bar{x} \right) \left[ \bigwedge_{i \leq n} p_i(\bar{x}) \approx q_i(\bar{x}) \right] \rightarrow p(\bar{x}) \approx q(\bar{x})$$.
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A quasivariety is finitely based provided it is definable by finitely many quasi-identities.
Relative congruences

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Theorem (A. I. Mal’cev, S. Burris)

Each algebra in a quasivariety $\mathcal{R}$ is isomorphic to a subdirect product of $\mathcal{R}$-subdirectly irreducible algebras.
Pigozzi theorem

A quasivariety $\mathcal{R}$ is \textit{relatively congruence-distributive} if for all $A$ in $\mathcal{R}$ the lattice of its $\mathcal{R}$-congruences is distributive.
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**Theorem (D. Pigozzi)**

A finitely generated relatively congruence-distributive quasivariety is finitely based.
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A variety version of Pigozzi theorem was proved earlier by K. Baker.
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There are finite algebras which generate congruence-distributive varieties and non relatively congruence-distributive quasivarieties.
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Definable $\mathcal{R}$-congruences

For a pair of elements $a, b$ of algebra $A$ let $\theta_{\mathcal{R}}(a, b)$ be the smallest $\mathcal{R}$-congruence of $A$ gluing $a$ and $b$. 
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Definition

1. An $\mathcal{R}$-congruence formula is a positive formula $\Gamma$ such that

   $\mathcal{R} \models (\forall x, u, v)[\Gamma(u, v, x, x) \rightarrow u \approx v]$. 
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2. A quasivariety $\mathcal{R}$ has **definable relative principal congruences** if there exists an $\mathcal{R}$-congruence formula $\Gamma$ such that

   \[
   \theta_\mathcal{R}(a, b) = \{(c, d) \in A^2 \mid A \models \Gamma(c, d, a, b)\}
   \]

   for all $a, b \in A \in \mathcal{R}$.
Definable $\mathcal{R}$-congruences, continued

Theorem (J. Czelakowski, W. Dziobiak)

The quasivariety $\mathcal{R}$ with definable relative principal congruences is finitely based iff the class $\mathcal{R}_{SI}$ of $\mathcal{R}$-subdirectly irreducible algebras is strictly elementary.
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Proof of if direction.

Let $\Gamma(x, y, u, v)$ define principal congruences in $\mathcal{R}$. 

There exists a finite set of quasi-identities $\Sigma$ such that $\mathcal{R} = \text{Mod}(\Sigma) \cap H(\mathcal{R})$ and $\mathcal{R}_{SI} = \text{Mod}(\Sigma)_{SI} \cap \mathcal{R}$.

There is a formula $\Delta(u, v)$ such that $A |_A = \Delta(c, d)$ iff $\{(e, f) \in A^2 |_A = \Gamma(e, f, c, d)\}$ is a $\text{Mod}(\Sigma)$-congruence of $A$ containing $(c, d)$. 

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2. There is a formula $\Delta(u, v)$ such that $A \models \Delta(c, d)$ iff

$$\{(e, f) \in A^2 \mid A \models \Gamma(e, f, c, d)\}$$

is a Mod($\Sigma$)-congruence of $A$ containing $(c, d)$. 
3 Because $\mathcal{R} \models (\forall u, v)\Delta(u, v)$, there is $I$ a finite set of identities such that

$$\mathcal{R} \models I \quad \text{and} \quad I \cup \Sigma \models (\forall u, v)\Delta(u, v).$$
Proof continued.

3. Because $\mathcal{R} \models (\forall u, v) \Delta(u, v)$, there is $I$ a finite set of identities such that

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4. Let

$$\psi = (\exists u, v) \left[ u \not\approx v \land (\forall x, y) \left[ x \not\approx y \rightarrow \Gamma(u, v, x, y) \right] \right]$$

and $\mathcal{R}_{SI} = \text{Mod}(\chi)$. 
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and $\mathcal{R}_I = \text{Mod}(\chi)$. Then there is a finite set $J$ of identities such that $\mathcal{R} \models J$ and $\Sigma \cup J \cup \{\psi\} \models \chi$. 

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and $\mathcal{R}_{SI} = \text{Mod}(\chi)$. Then there is a finite set $J$ of identities such that $\mathcal{R} \models J$ and $\Sigma \cup J \cup \{\psi\} \models \chi$.

5. We have $R_{SI} = \text{Mod}(\Sigma \cup I \cup J)_{SI}$ and thus $R = \text{Mod}(\Sigma \cup I \cup J)$. 

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Definable relative principal subcongruences

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Relative congruence-distributive quasivariety does not need to have definable relative principal congruences.
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Definition

A quasivariety $\mathcal{R}$ has **definable relative principal subcongruences** if there are $\mathcal{R}$-congruence formulas $\Gamma_1, \Gamma_2$ such that for all $A \in \mathcal{R}$ and each pair of distinct elements $a, b \in A$, there is a pair of distinct elements $c, d \in A$ such that

$$A \models \Gamma_1(c, d, a, b) \quad \text{and} \quad \theta_{\mathcal{R}}(c, d) = \{(e, f) \mid A \models \Gamma_2(e, f, c, d)\}.$$
Fact (Mostly due to K. Baker and J. Wang)

A finitely generated relatively congruence-distributive quasivariety has definable relative principal subcongruences.
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Fact

The quasivariety $\mathcal{R}$ with definable relative principal subcongruences is finitely based iff the class $\mathcal{R}_{SI}$ of $\mathcal{R}$-subdirectly irreducible algebras is strictly elementary.
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The quasivariety $\mathcal{R}$ with definable relative principal subcongruences is finitely based iff the class $\mathcal{R}_{SI}$ of $\mathcal{R}$-subdirectly irreducible algebras is strictly elementary.

Proof.

By refining the proof of Czelakowski-Dziobiak theorem. \qed
Proof of Pigozzi theorem, continued

Proof of Pigozzi theorem.

Let $\mathcal{F}$ be a finite family of finite algebras and $\mathcal{R} = \text{SP}(\mathcal{F})$. Then $\mathcal{R}_{SI} \subseteq S(\mathcal{F})$ is a finite family of finite algebras and hence it is strictly elementary. Now use previous facts.
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Let $\mathcal{F}$ be a finite family of finite algebras and $\mathcal{R} = \text{SP}(\mathcal{F})$. Then $\mathcal{R}_{SI} \subseteq \mathcal{S}(\mathcal{F})$ is a finite family of finite algebras and hence it is strictly elementary. Now use previous facts.

Problem

Let $\mathcal{R}$ be a relatively congruence-distributive quasivariety and assume that the class $\mathcal{R}_{SI}$ of $\mathcal{R}$-subdirectly irreducible algebras is strictly elementary. Must $\mathcal{R}$ be finitely based?
Thank you for your attention :-}