ItMM 4 Functions

4.1 Which of the following relations are functions? Find their domains and sets of values. Which of them are injections?

a) for $x, y \in \mathbb{R}$, $xRy \Leftrightarrow x^3 = y^4$,

b) for $x, y \in \mathbb{R}$, $xRy \Leftrightarrow \frac{y-1}{x} = 1$,

c) for $x, y, z \in \mathbb{R}$, $(x, y)Sz \Leftrightarrow x + y + z^2 = 1$

d) for $x, y \in \mathbb{N}$, $xUy \Leftrightarrow x$ is the greatest prime divisor of y,

f) for polynomial p and $x \in \mathbb{R}$, $pTx \Leftrightarrow p(x) = 0$,

g) for $A, B \subseteq X$, $A\mathcal{F}B \Leftrightarrow A \cup B = X$ *and* $A \cap B = \emptyset$,

h) for $A \subseteq \mathbb{N}, x \in \mathbb{N}$, $A\mathcal{G}x \Leftrightarrow x$ is a product of all elements of A,

i) for a quadratic polynomial f (with real coefficients) and $A \subseteq \mathbb{R}$, $f\mathcal{V}A \Leftrightarrow (\forall x \in \mathbb{R})x \in A \Leftrightarrow f(x) = 1$,

j) for $x, y \in \mathbb{R}$, $xVy \Leftrightarrow (\exists z \in \mathbb{R})y = \frac{x+z}{2}$

4.2 Let X be a finite set and $f : X \to X$. Prove that f is one-to-one if and only if f is "onto".

4.3 Let $A \subseteq X$ and $\chi_A : X \to \{0, 1\}$ be a characteristic function of a set A i.e. for $x \in X$, $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$. Prove that the function F defined $F(A) = \chi_A$ is a on-to-on mapping of the powerset of X onto set $\{0, 1\}^X$.

4.4 Let $A, B \subseteq X$ and $x \in X$. Prove

a) $\chi_{-A}(x) = 1 - \chi_A(x)$,

b) $\chi_{A \cap B}(x) = \chi_A(x) \cdot \chi_B(x)$.

4.5 For a function f and a subset A of its domain find $f(A)$ and $f^{-1}(f(A))$

a) $f(x) = x^2$ for $x \in \mathbb{R}$, $A = [-2; 3)$,

b) $f(x, y) = x + y$ for $x, y \in \mathbb{R}$, $A = \{(0, 0), (0, 1)\}$.

c) $f(x, y) = \frac{x}{y+1}$ for $x, y \in \mathbb{N}$, $A = \{(1, 2)\}$.

d) $f(p) = p(1)$ for a polynomial p with real coefficients, A is the family of all linear functions of the form $ax - a$.

e) $f(x, y) = (x + y, x - y)$ for $x, y \in \mathbb{R}$, $A = \{(x, y) \in \mathbb{R}^2 : x = y\}$.

f) $f(x, y) = x^2 + y^2$ for $x, y \in \mathbb{R}$, $A = \{(x, y) \in \mathbb{R}^2 : 1 < x < 2 \wedge 0 < y < 1\}$.

g) $f(x, y) = max(x, y)$ for $x, y \in \mathbb{R}$, $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

h) $f(n) =$ sum of all prime divisors of n for $n \in \mathbb{N}$, $A = \{4, 6\}$,

i) $f(X) = X \times X$ for $X \subseteq \mathbb{R}$, $A = \{[-x; x] : x \in \mathbb{R}\}$,

j) $f(X) = \{x \in \mathbb{R} : (\exists y)(x, y) \in X\}$ for $X \subseteq \mathbb{R}^2$, A is the family of all singletons.

4.6 Which inclusion is true? $f(A) \cap f(B) \subseteq f(A \cap B)$ or $f(A \cap B) \subseteq f(A) \cap f(B)$? Prove the true one or find a counterexample for the false one. Prove that if f is a bijection then both are true.

4.7 Which inclusion is true? $f(A) - f(B) \subseteq f(A - B)$ or $f(A - B) \subseteq f(A) - f(B)$? Prove the true one or find a counterexample for the false one. Prove that if f is a bijection then both are true.

4.8 Let X be a finite set and $f : X \to X$. Prove that there exists $A \subseteq X$ such that $f(A) = A$.

4.9 Write the mathematical formulas corresponding to the following statements with the help of the following signs only: propositional connectives, quantifiers, variables varied through set $\mathbb{R}$ and symbols $\in, \mathbb{R}, \mathbb{R}^{\mathbb{R}}, \leqslant, <, =, \cdot, +, -$.

a) function f is increasing,

b) function f is increasing or decreasing,

c) function f is bounded,.

d) function f has maximum,

e) function f has exactly one maximum,

f) function f has maximum or minimum,

g) If a has maximum than it is bounded from above,

h) increasing function has no maximum nor minimum

i) increasing function is one-to-one,

j) function bounded from above may have no maximum,

k) there is no even increasing function.